<u>Last time:</u> Representation theorems.

<u>Corollary:</u> Let $M \in \mathcal{M}_2^{cont}$ be adapted to $\{\tilde{\mathcal{F}}_t^B\}$ where $B = SBM$ and $\tilde{\mathcal{F}}_t^B := \sigma(\{B_s : s \le t\} \cup \mathcal{N})$. Then $\exists\, Y \in \mathcal{V}_B$ s.t.

$$\forall t \ge 0: \quad M_t = M_0 + \int_0^t Y_s \, dB_s.$$

Pf In Thm: $X \in L^2 \Rightarrow \forall t \ge 0: E(X \mid \tilde{\mathcal{F}}_t) = EX_0 + \int_0^t Y_s \, dB_s$ for some $Y \in \mathcal{V}_B$.

Pick $n \in \mathbb{N}$, $n \ge 1$: Then $M_t = E(M_n \mid \tilde{\mathcal{F}}_t^B)$ $\forall t \in [0,n]$.

So by Thm: $\exists\, Y^{(n)} \in \mathcal{V}_B$: $M_t = E(M_n \mid \tilde{\mathcal{F}}_t^B) = EM_t + \int_0^t Y_s^{(n)} \, dB_s$.

Uniqueness: $\forall m \ge n$: $\lambda(\{t \in [0,n] : Y_t^{(n)} \ne Y_t^{(m)}\}) = 0$ a.s.

Define: $Y_t := \sum_{n \ge 1} Y_t^{(n)} 1_{[n-1,n)}(t)$. Then claim holds with Y.

Since $EM_t = EM_0 = M_0$ a.s. by $P|_{\tilde{\mathcal{F}}_0^B}$ is trivial, we are done. $\boxtimes$

Girsanov's Thm

- exponential change of measure, "tilting".

Lemma Let $X_1 \dots X_n$ be independent RVs s.t. $\forall \lambda \in \mathbb{R}$: $\varphi_i(\lambda) := E(e^{\lambda X_i}) < \infty$.
For $\underline{\lambda} \in \mathbb{R}^n$, $A \in \sigma(X_1 \dots X_n)$, set

$$P_{\underline{\lambda}}(A) := E\left(1_A \exp\left\{\sum_{i=1}^n \left[\lambda_i X_i - \log \varphi_i(\lambda_i)\right]\right\}\right) \qquad \underline{\lambda} = (\lambda_1 \dots \lambda_n)$$

Then $P_{\underline{\lambda}}$ is prob. law s.t. $X_1 \dots X_n$ are still independent but $\forall i = 1 \dots n$: $E_{\underline{\lambda}}(X_i) = \dfrac{\varphi_i'(\lambda_i)}{\varphi_i(\lambda_i)}$

Pf Suffices to prove for $n=1$.

$$E\left(X e^{\lambda X - \log \varphi(\lambda)}\right) = \frac{1}{\varphi(\lambda)} \frac{d}{d\lambda} E\left(e^{\lambda X}\right) = \frac{\varphi'(\lambda)}{\varphi(\lambda)}. \qquad \boxtimes$$

Exponential "tilting" is a technique to change the mean.

Used widely in large-deviations theory.

<u>Lemma</u> (Tilted Gaussians) Let $X = (X_1 \dots X_n)$ be multivariate normal with $EX = 0$ and $C := \{C(X_i, X_j)\}_{i,j=1}^n$ general.

For $\lambda \in \mathbb{R}^n$ set

$$P_\lambda(A) := E\left(1_A e^{\lambda \cdot X - \frac{1}{2}\lambda \cdot C\lambda}\right)$$

Then

$$X - C\lambda \text{ under } P_\lambda \overset{\text{law}}{=} X \text{ under } P$$

Pf: Pick $t \in \mathbb{R}^n$ and calculate

$$E_\lambda\left(e^{t\cdot(X - C\lambda)}\right) = E\left(e^{t\cdot(X-C\lambda) + \lambda\cdot X - \frac{1}{2}\lambda\cdot C\lambda}\right)$$

$$= \underbrace{E\left(e^{(t+\lambda)\cdot X - \frac{1}{2}(t+\lambda)\cdot C(t+\lambda)}\right)}_{=1} e^{\frac{1}{2}t\cdot Ct}$$

$$= e^{\frac{1}{2}t\cdot Ct} = E\left(e^{t\cdot X}\right) \qquad \boxtimes$$

Curtiss + Cramér-Wold then give the claim.

Lemma Let $X_1, \ldots X_n$ be independent with $X_i = \mathcal{N}(0, \sigma_i^2)$.

Let $\lambda_k : \mathbb{R}^{k-1} \to \mathbb{R}$ be (bdd) Borel for each $k = 1, \ldots, n$. (λ_1 is a constant).

Let $P_\lambda(A) := E\left(1_A \exp\left\{ \sum_{k=1}^{n} \lambda_k(X_1 \ldots X_{k-1}) X_k - \frac{1}{2} \sum_{k=1}^{n} \lambda(X_1 \ldots X_{k-1})^2 \sigma_k^2 \right\}\right)$

Then P_λ is a prob. measure and

$$\left\{ X_k - \sum_{j=1}^{k} \lambda_j(X_1 \ldots X_{j-1}) \sigma_j^2 \right\}_{j=1}^{n} \text{ under } P_\lambda \overset{\text{law}}{=} X \text{ under } P.$$

Pf: Let $M_k := \prod_{j=1}^{k} e^{\lambda(X_1 \ldots X_{j-1}) X_j - \frac{1}{2}\lambda(X_1 \ldots X_{j-1})^2 \sigma_j^2}$.

$\mathcal{F}_k = \sigma(X_1 \ldots X_k)$. Then $E(M_k \mid \mathcal{F}_{k-1}) = M_{k-1}$ a.s.

So $\{M_k\}_{k=1}^{n}$ is martingale w.r.t. $\{\mathcal{F}_k\}_{k=1}^{n}$. So $E M_n = E M_0 = 1$

and P_λ is thus a probability measure.

Denote $\tilde{X}_k := X_k - \sum_{j=1}^{k} \lambda(X_1 \ldots X_{j-1}) \sigma_j^2$

Pick $t \in \mathbb{R}^n$ and set

$$N_k = \prod_{j=1}^{k} e^{t_i \tilde{X}_i - \frac{1}{2} t_i^2 \sigma_i^2}$$

Then

$$N_k M_k = \prod_{j=1}^{k} e^{(t_j + \lambda_j(X_1 \dots X_{j-1}))X_j - \frac{1}{2}[t_j + \lambda_j(X_1 \dots X_{j-1})]^2 \sigma_j^2}$$

So $\{N_k M_k\}_{k=1}^{n}$ is again a martingale. So previous argument.

$$1 = E(N_0 M_0) = E(N_n M_n) = E_\lambda(N_n)$$ so claim follows from Curtiss Thm. $\boxtimes$

Thm (Girsanov '60) Let $B = SBM$, $Y \in \mathcal{V}_B^{loc}$.

Define $M_t := \exp\left\{ \int_0^t Y_s \, dB_s - \frac{1}{2} \int_0^t Y_s^2 \, ds \right\}$

Then $M \in \mathcal{M}_{loc}^{cont}$. If $E(M_t) = 1$, then $\{M_{s \wedge t} : s \geq 0\}$ is a martingale and, $\tilde{P}(A) := E(1_A M_t)$ defines a prob. measure on $\mathcal{F}_t$.

Moreover $\left\{ \tilde{B}_s := B_s - \int_0^s Y_u \, du : s \in [0,t] \right\}$ is SBM under $\tilde{P}$.