Representation theorems

last time: Lévy characterization $M \in \mathcal{M}_{loc}^{cont}$, $\langle M \rangle_t = t \Rightarrow M = SBM$.

true even if M vector valued ... $\langle M_t^{(i)}, M_t^{(j)} \rangle = t \text{ id}$

If not true
- make it true by time change $\longleftarrow$ multiple of id
- represent M as stoch integral w.r.t SBM.

Thm Let $M \in \mathcal{M}_{loc}^{cont}$ be s.t. $t \longmapsto \langle M \rangle_t$ is AC a.s.
Extending the probability space so that it contains an independent SBM,
there exists a SBM $\{ B_t : t \geq 0 \}$ s.t.

$$\forall t \geq 0 : \ M_t = \int_0^t \sqrt{\frac{d\langle M \rangle_s}{ds}} \, dB_s.$$

Pf: Let $\{\hat{B}_t : t \geq 0\}$ be SBM $\perp\!\!\!\perp$ $\{M_t : t \geq 0\}$.

Augment the filtration into $\tilde{\mathcal{F}}_t = \sigma(\mathcal{F}_t \cup \sigma(\hat{B}_s : s \leq t) \cup \underbrace{\mathcal{N}}_{\text{null sets}})$

Define

$$B_t := \int_0^t \left(\frac{d\langle M\rangle_s}{ds}\right)^{-1/2} 1_{\frac{d\langle M\rangle_s}{ds} > 0}\, dM_s + \int_0^t 1_{\frac{d\langle M\rangle_s}{ds} = 0}\, d\tilde{B}_s$$

where we choose continuous versions of stoch. integrals on RHS.

Then

$$\langle B\rangle_t = \int_0^t \left(\frac{d\langle M\rangle_s}{ds}\right)^{-1} 1_{\frac{d\langle M\rangle_s}{ds} > 0}\, d\langle M\rangle_s + \int_0^t 1_{\frac{d\langle M\rangle_s}{ds} = 0}\, ds + \underbrace{\text{cross variation}}_{\overset{\text{by } M \perp\!\!\!\perp \tilde{B}}{=}\ 0}$$

Writing $d\langle M\rangle_s = \frac{d\langle M\rangle_s}{ds}\, ds$, gives $\langle B\rangle_t = t$.

Lévy characterization: B = SBM.

Then

$$\int_0^t \left(\frac{d\langle M\rangle_s}{ds}\right)^{1/2} dB_s \overset{\text{Substitution rule for Itô integral}}{=} \int_0^t \left(\frac{d\langle M\rangle_s}{ds}\right)^{1/2} \left(\left(\frac{d\langle M\rangle_s}{ds}\right)^{-1/2} 1_{\frac{d\langle M\rangle_s}{ds} > 0}\, dM_s + 1_{\frac{d\langle M\rangle_s}{ds} = 0}\, d\hat{B}_s \right)$$

$$= \int_0^t 1_{\frac{d\langle M\rangle_s}{ds} > 0}\, dM_s \underset{\int_0^t \left(1_{\frac{d\langle M\rangle}{ds}} - 1\right)^2 d\langle M\rangle_s = 0}{=} \int_0^t 1\, dM_s = M_t \quad \text{a.s.}$$

$\boxed{\times}$

Lemma (Sub-rule for Itô) Let $M \in \mathcal{M}^{cont}_{loc}$, $X \in \mathcal{V}^{loc}_M$, Y s.t. $XY \in \mathcal{V}^{loc}_M$.

Define $N_t = \int_0^t X_s \, dM_s$ (cont. version). Then $N \in \mathcal{M}^{cont}_{loc}$ and we have $Y \in \mathcal{V}^{loc}_N$ and

$$\forall t \geq 0: \quad \int_0^t Y_s \, dN_s = \int_0^t Y_s X_s \, dM_s.$$

In short: $dN_s = X_s \, dM_s$.

Pf Assume first: $Y_s = Y_v \mathbb{1}_{(v,u]}(s)$. Then

$$\int_0^t X_s Y_s \, dM_s = Y_u \left(\int_0^{t\wedge u} X_s \, dM_s - \int_0^{t \wedge v} X_s \, dM_s \right)$$

$$= Y_u (N_{t\wedge u} - N_{t \wedge v}) = \int_0^t Y_s \, dN_s$$

Linearity extends this to $Y \in \mathcal{V}_0$. Now take $\{Y^{(n)}\} \in \mathcal{V}_0^{\mathbb{N}}$ s.t. $\int_0^t (Y_s^{(n)} - Y_s)^2 \, d\langle N \rangle_s \xrightarrow[n\to\infty]{P} 0$

Since $d\langle N \rangle_s = X_s^2 \, d\langle M \rangle_s$, we get $\int_0^t (X_s Y_s^{(n)} - X_s Y_s)^2 \, d\langle M \rangle_s \xrightarrow[n\to\infty]{P} 0$

Then $\int_0^t Y_s^{(n)} \, dN_s \xrightarrow{P} \int_0^t Y_s \, dN_s$ and $\int_0^t X_s Y_s^{(n)} \, dM_s \xrightarrow{P} \int_0^t X_s Y_s \, dM_s$. $\boxtimes$

Brownian martingales:

Thm Let $B = SBM$, $\mathcal{F}_t^B := \sigma(B_s : s \le t)$ $\left(\text{even } \tilde{\mathcal{F}}_t := \sigma(\mathcal{F}_t \cup \mathcal{N})\right)$

Let $X \in L^2$. Then $\exists \{Y_s : s \ge 0\} \in \mathcal{V}_B$ s.t.

$\forall t \ge 0$: $$E(X \mid \mathcal{F}_t^B) = E(X) + \int_0^t Y_s \, dB_s \quad \text{a.s.}$$

The process Y is determined uniquely as an element of $\mathcal{V}_B$.

Pf: Construction:

$$\inf \left\{ E\left(\left(E(X \mid \mathcal{F}_t) - \int_0^t Y_s \, dB_s \right)^2 : Y \in \mathcal{V}_B \right) \right.$$

is achieved by taking minimizing sequence $\{Y^{(n)}\}$ and using parallelogram law to prove that $\int_0^t Y^{(n)} dB_s$ converges in L^2. Then $\{Y^{(n)}\}$ converge in $L^2([0,t] \times \Omega)$ by Itô isometry.

Writing Y for limit process (defined only up to time t) we have

$$E\left(\left[E(X \mid \mathcal{F}_t^B) - \int_0^t Y_s \, dB_s \right] \int_0^t Z_s \, dB_s \right) = 0 \quad \forall Z \in \mathcal{V}_B.$$

checking the identity: Denote $V_t := E(X|\mathcal{F}_t^B) - \int_0^t Y_s\, dB_s$.

For Z choose: $Z_s := \left(i\sum_{j=1}^n \lambda_j 1_{(t_{j-1},t_j]}^{(s)}\right) \underbrace{\exp\left\{ i\sum_{j=1}^n \lambda_j (B_{t_j\wedge s} - B_{t_{j-1}\wedge s}) + \frac{1}{2}\sum_{j=1}^n \lambda_j^2 (t_j\wedge s - t_{j-1}\wedge s)\right\}}_{M_s}$

Then $dM_s = Z_s\, dB_s$.

So $\int_0^t Z_s\, dB_s = M_t - M_0 = M_t - 1$

WLOG: assume $EX = 0$

When we clean up, we get $E\left(V_t\, e^{i\sum_{j=1}^n \tilde{\lambda}_j B_{t_j}}\right) = 0$

for any $0 = t_0 < t_1 < \dots < t_n < t$

$\tilde{\lambda}_1, \dots, \lambda_n \in \mathbb{R}$.

So $E(V_t \mid \sigma(B_{t_1} \dots B_{t_n})) = 0$ a.s.

Levy forward thm:

$E(V_t \mid \mathcal{F}_t^B) = 0$ a.s.

$\Rightarrow V_t = 0$ a.s. $\boxtimes$