## Relating cont. loc. martingales to Brownian motion

**last time:** $M \in \mathcal{M}_2^{\text{cont}}$: $\int_0^t Y_s \, dM_s$ defined for $Y \in \overline{\mathcal{V}_0}^{[\cdot]_M}$.

$\langle M \rangle$ AC $\Rightarrow \overline{\mathcal{V}_0}^{[\cdot]_M} = \mathcal{V}_M$.

$\langle M \rangle$ general $\Rightarrow \forall Y \in \mathcal{V}_M$: progressively meas $\Rightarrow Y \in \overline{\mathcal{V}_0}^{[\cdot]_M}$.

**Lemma:** Suppose $\mathcal{F}_0 \supseteq$ P-null sets. Then $\forall M \in \mathcal{M}_2^{\text{cont}}$:

$$Y \in \overline{\mathcal{V}_0}^{[\cdot]_M} \iff \exists \hat{Y} \in \mathcal{V}_M: \text{ progressively meas} \wedge \int_0^\infty 1_{Y_t \neq \tilde{Y}_t} \, d\langle M \rangle_t = 0.$$

## Extension to local martingales & loc. integrable processes.

**Thm** Let $M \in \mathcal{M}_{\text{loc}}^{\text{cont}}$, $Y \in \mathcal{V}_M^{\text{loc}} := \{ \tilde{Y} : \text{adapted}, \langle \text{meas.}, \text{progr. meas} \rangle, \forall t \geq 0: \int_0^t \tilde{Y}_s^2 \, d\langle M \rangle_s < \infty \text{ a.s.} \}$

Set $\tau_K := \inf \{ t \geq 0 : \langle M \rangle_t \geq K \vee \int_0^t Y_s^2 \, d\langle M \rangle_s \geq K \}$.

Then $\forall \tilde{K} \geq K$: $\int_0^t Y_s 1_{\tau_{\tilde{K}} > s} \, dM_{s \wedge \tau_{\tilde{K}}} = \int_0^t Y_s 1_{\tau_K > s} \, dM_{s \wedge \tau_K}$ a.s. $\{ \tau_K > t \}$.

Then $\int_0^t Y_s \, dM_s := \lim_{K \to \infty} \int_0^t Y_s 1_{\tau_K > s} \, dM_{\tau_K \wedge s}$ exists a.s. $\forall t \geq 0$.

Moreover, assuming $\mathcal{F}_0 \supseteq$ P-null sets, then exists a continuous process $I(Y) = \{I_t(Y) : t \geq 0\}$, s.t.

$$\forall t \geq 0: \quad I_t(Y) = \int_0^t Y_s \, dM_s \quad \text{a.s.}$$

and $I(Y) \in \mathcal{M}^{\text{cont}}_{\text{loc}}$ with $\langle I(Y) \rangle_t = \int_0^t Y_s^2 \, d\langle M \rangle_s$ a.s.

Finally, for all stopping times $T$,

$$\forall t \geq 0: \quad \int_0^{T \wedge t} Y_s \, dM_s := I_{T \wedge t}(Y) = \int_0^t Y_s \mathbb{1}_{T > s} \, dM_s .$$

Correct way to approximate loc. integrals is in $L^2$/prob-sense:

<u>Lemma</u> $\forall M \in \mathcal{M}^{\text{cont}}_{\text{loc}}$, $Y, Y^{(n)} \in \mathcal{V}^{\text{loc}}_M$:

$$\int_0^t (Y_s - Y_s^{(n)})^2 \, d\langle M \rangle_s \xrightarrow[n \to \infty]{P} 0 \implies \int_0^t Y_s^{(n)} \, dM_s \xrightarrow[n \to \infty]{P} \int_0^t Y_s \, dM_s .$$

<u>Lemma</u> $\forall M \in \mathcal{M}^{\text{cont}}_{\text{loc}}$, $Y \in \mathcal{V}^{\text{loc}}_M$ s.t. $Y$ is continuous. Then $\forall t \geq 0$ $\forall \{\Pi_n\}$ partitions of $[0,t]$ s.t. $\Pi_n = \{0 = t_0^n < \dots < t_{m_n}^n = t\}$ with $\|\Pi_n\| \to 0$:

$$\sum_{i=1}^{m_n} Y_{t_{i-1}^n} (M_{t_i^n} - M_{t_{i-1}^n}) \xrightarrow[n \to \infty]{P} \int_0^t Y_s \, dM_s .$$

**Def** A <u>semimartingale</u> is a process of the form $\{M_t + A_t : t \geq 0\}$ where $\{M_t : t \geq 0\}$ is loc. martingale and $\{A_t : t \geq 0\}$ is adapted, measurable and s.t. $\forall t \geq 0$: $V_t^{(1)}(A) < \infty$ and $A_0 = 0$.

**Note** For continuous semimartingales the decomposition is unique up to indistinguishability.

Integrals defined then as: $$\int_0^t Y_s \, dX_s \underset{X_t = M_t + A_t}{=} \int_0^t Y_s \, dM_s + \int_0^t Y_s \, dA_s$$ assuming these exist.

Also: $\langle X \rangle_t := \langle M \rangle_t$.

**Thm** (Itô) $\forall f \in C^2(\mathbb{R})$ $\forall X$ = cont. semimartingale:

$$\forall t \geq 0: \quad f(X_t) = f(X_0) + \int_0^t f'(X_s) \, dX_s + \frac{1}{2} \int f''(X_s) \, d\langle X \rangle_s )$$

**Pf** Same as for SBM.

**Thm** (Lévy characterization of SBM) Let $M \in \mathcal{M}^{cont}_{loc}$ be s.t. $\forall t \geq 0: \langle M \rangle_t = t$. Then $M$ is a standard Brownian motion

Pf: $Z_t := e^{i\lambda M_t + \frac{\lambda^2}{2}\langle M \rangle_t}$. Enhanced Itô formula:

$$dZ_t = Z_t\left(i\lambda\, dM_t + \frac{\lambda^2}{2} d\langle M \rangle_t\right) + \frac{1}{2} Z_t (i\lambda)^2 d\langle M \rangle_t = i\lambda Z_t\, dM_t.$$

So $Z \in \mathcal{M}^{cont}_{loc}$. If bounded, $Z \in \mathcal{M}^{cont}$.

Hence, $\forall s<t$: $E(Z_t | \mathcal{F}_s) = Z_s$ a.s.

For $\langle M \rangle_t = t$, this translates into:

$$E\left(e^{i\lambda M_t + \frac{\lambda^2}{2}t} \,\middle|\, \mathcal{F}_s\right) = e^{i\lambda M_s + \frac{\lambda^2}{2}s} \quad \text{a.s.}$$

which turns into

$$E\left(e^{i\lambda(M_t - M_s)} \,\middle|\, \mathcal{F}_s\right) = e^{-\frac{\lambda^2}{2}(t-s)} \quad \text{a.s.}$$

So $\{M_t : t \geq 0\}$ has FDD's of SBM. Being continuous, it is a SBM. $\boxtimes$

**Thm** Let $M \in \mathcal{M}^{\text{cont}}_{\text{loc}}$ be s.t. $t \longmapsto \langle M \rangle_t$ is AC and strictly increasing a.s.
Then there exists a SBM $\{B_t : t \geq 0\}$, adapted (same filtration)
s.t.

$$\forall t \geq 0; \quad M_t = \int_0^t \sqrt{\frac{d\langle M \rangle_s}{ds}}\, dB_s. \quad \text{a.s.}$$

**Idea:**

$$B_t = \int_0^t \left( \frac{d\langle M \rangle_s}{ds} \right)^{-1/2} 1_{\{\frac{d\langle M \rangle_s}{ds} \neq 0\}}\, dM_s.$$

$$\langle B \rangle_t = \int_0^t \left( \frac{d\langle M \rangle_s}{ds} \right)^{-1} 1_{\frac{d\langle M \rangle_s}{ds} \neq 0}\, d\langle M \rangle_s = t.$$

Now need change of variables inside Itô integral.

What if AC not true?

Need also $\lim_{t \to \infty} \langle M \rangle_t = \infty$

**Thm** (Dambis '62, Dubins-Schwarz '65)
Let $M \in \mathcal{M}^{\text{cont}}_{\text{loc}}$. Define $T(t) := \inf\{ s \geq 0 : \langle M \rangle_s > t \}$.
Then $\{ M_{T(t)} : t \geq 0 \}$ is SBM with respect to $\{ \mathcal{F}_{T(t)} : t \geq 0 \}$,
and setting $B_t := M_{T(t)}$, we have

$$\forall t \geq 0; \quad M_t = B_{\langle M \rangle_t} \quad \text{a.s.}$$