Wrapping up Itô integral wrt continuous local martingales

last time: $M \in \mathcal{M}_2^{cont}$, $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M} \longmapsto \int_0^t Y_s \, dM_s$

Itô isometry $E\left[\left(\int_0^t Y_s \, dM_s\right)^2\right] = E\left(\int_0^t Y_s^2 \, d\langle M\rangle_s\right)$

$\langle \int_0^\cdot Y_s \, dM_s \rangle_t = \int_0^t Y_s^2 \, d\langle M\rangle_s.$

Q: What is $\overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}$?

Prop Suppose $M \in \mathcal{M}_2^{cont}$ is s.t. $t \longmapsto \langle M\rangle_t$ is AC a.s.
Then $\overline{\mathcal{V}_0}^{[\![\cdot]\!]_M} = \mathcal{V}_M := \{ Y : \text{adapted, measurable}, \forall t \geq 0 : E \int_0^t Y_s^2 \, d\langle M\rangle_s < \infty \}$

Pf Suppose $Y \in \mathcal{V}_M$ is bounded, $\forall t \geq 0$: $|Y_t| \leq K$.
A lemma from 275 D: $\exists \{Y^{(n)}\} \in \mathcal{V}_0$ s.t.

$$\forall t \geq 0 : E \int_0^t (Y_s - Y_s^{(n)})^2 \, ds \xrightarrow[n \to \infty]{} 0$$

Choosing subsequences, we may assume pointwise convergence a.e.

$$\lambda\left(\left\{t\ge 0 : \limsup_{n\to\infty} |Y_t - Y_t^{(n)}| > 0\right\}\right) = 0 \quad \text{a.s.} \qquad \left(\text{wlog } |Y_\cdot^{(n)}| \le K\right)$$

Then

$$\int_0^t (Y_s - Y_s^{(n)})^2 \, d\langle M\rangle_s \overset{AC}{=} \int_0^t \underbrace{(Y_s - Y_s^{(n)})^2}_{\le 4K^2} \frac{d\langle M\rangle_s}{ds} \, ds \xrightarrow[n\to\infty]{DCT} 0 \quad \text{a.s.}$$

DCT on LHS:

$$\forall t\ge 0: \; E \int_0^t (Y_s - Y_s^{(n)})^2 \, d\langle M\rangle_s \longrightarrow 0 \Rightarrow [\![Y - Y^{(n)}]\!]_M \to 0$$

$$\Rightarrow Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M} \text{ if } Y \in \mathcal{V}_M \text{ is bounded}$$

If Y is unbounded, set

$$Y_t^{(n)} := Y_t \mathbb{1}_{|Y_t| \le n} \text{ . Then } Y^{(n)} \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}.$$

Then

$$E\left(\int_0^t (Y_s - Y_s^{(n)})^2 d\langle M\rangle_s\right) = E\left(\int_0^t \underbrace{Y_s^2 \mathbb{1}_{|Y_s| > n} \, d\langle M\rangle_s}_{\xrightarrow[n\to\infty]{} 0 \text{ by BCT}}\right) \xrightarrow[n\to\infty]{\substack{(Y\in\mathcal{V}_M)\\ DCT}} 0$$

So $Y \in \overline{\mathcal{V}}^{[\![\cdot]\!]_M}$ as well. $\boxtimes$

Prop Let $M \in \mathcal{M}_2^{cont}$ and assume $Y \in \mathcal{V}_M$ is progressively measurable.
Then $Y \in \overline{\mathcal{V}_0}^{[[\cdot]]_M}$.

Def $\{Y_s : s \geq 0\}$ is progressively meas. w.r.t. $\{\mathcal{F}_t\}$ if

$\forall t \geq 0$: $(\omega, s) \mapsto Y_s(\omega)$ is $\mathcal{F}_t \otimes \mathcal{B}([0,t]) / \mathcal{B}(\mathbb{R})$-measurable.

Stronger than just measurability?

T stopping time, $\mathcal{F}_T = \dots$, X_T is $\mathcal{F}_T$-meas if X is progressively measurable

$\Omega = \mathbb{R}_+$, $\omega \in \mathbb{R}_+$

$X_t^{(\omega)} = 1_{\{\omega\}}(t)$

$\mathcal{F}_t = \sigma(\{\{x\} : x \in \mathbb{R}_+\})$

Proof Key idea: Use time change w.r.t. $t \mapsto \langle M \rangle_t + t$.

$\forall u \geq 0$: $T(u) := \inf\{t \geq 0 : \langle M \rangle_t + t \geq u\}$.

Then $u \mapsto T(u)$ is continuous, strictly increasing, and obeys

$$\langle M \rangle_{T(u)} + T(u) = u$$

Assume WLOG: $\exists K \geq 0 \ \forall t \geq 0$: $|Y_t| \leq K$

Now let $Y \in \mathcal{V}_M$ be progressively measurable. Then

$\{Y_{T(u)} : u \ge 0\}$ is jointly measurable, adapted to $\{\mathcal{F}_{T(u)}\}_{u \ge 0}$.

Also, $\forall t \ge 0$

$$E\left(\int_0^t Y_{T(s)}^2\, ds\right) < \infty \quad \text{by } Y \text{ being bounded.}$$

new filtration

Lemma for 275D: $\exists \{Y^{(n)}\}_{n \in \mathbb{N}} \in \mathcal{V}_0$ of the form:

$$Y_u^{(n)} = Z_0^{(n)} 1_{\{0\}}(u) + \sum_{i=1}^{m_n} Z_i^{(n)} 1_{(t_{i-1}^n, t_i^n]}(u)$$

where Z_i is $\mathcal{F}_{T(t_{i-1}^n)\vee 0}$-measurable for $i = 0 \dots m_n$

s.t. $\forall t \ge 0$: $E\left(\int_0^t \left(Y_{T(s)} - Y_s^{(n)}\right)^2 ds\right) \xrightarrow[n \to \infty]{} 0$

Abbreviate $\tilde{Y}_t^{(n)} = Y^{(n)}_{\langle M\rangle_t + t}$ and note:

$$E\left(\int_0^t \left(Y_s - \tilde{Y}_s^{(n)}\right)^2 d\langle M\rangle_s\right) \le E\left(\int_0^{t+\cancel{\langle M\rangle_t}} \left(Y_s - Y^{(n)}_{\langle M\rangle_s + s}\right)^2 \left(d\langle M\rangle_s + ds\right)\right)$$

$$\cancel{\le} = E\left(\int_0^{t+\langle M\rangle_t} \left(Y_{T(u)} - Y_u^{(n)}\right)^2 du\right) \le E\left(\int_0^{t+\alpha} \left(Y_{T(u)} - Y_u^{(n)}\right)^2 du\right) + 4K^2 E\left(\left(t + \langle M\rangle_t\right) 1_{\langle M\rangle_t \ge \alpha}\right) \xrightarrow[\alpha \to \infty]{DCT} 0$$

Now $\tilde{Y}^{(n)}_u = Z^{(n)}_0 \mathbb{1}_{\{0\}}(u) + \sum_{i=1}^{n} Z^{(n)}_i \mathbb{1}_{(T(t^n_{i-1}), T(t^n_i)]}(u)$

Approximate: $T_k(s) := 2^{-k} \lceil 2^k T(s) \rceil$, then $T_k(s) \downarrow T(s)$

and so

$$\mathbb{1}_{(T_k(s), T_k(t)]}(u) \longrightarrow \mathbb{1}_{(T(s), T(t)]}(u)$$

So $\tilde{Y}^{(n,k)}_u := Z^{(n)}_0 \mathbb{1}_{\{0\}}(u) + \sum_{i=1}^{n} Z^{(n)}_i \mathbb{1}_{(T_k(t^n_{i-1}), T_k(t^n_i)]}(u)$

obey $\tilde{Y}^{(n,k)}_u \xrightarrow[k \to \infty]{} \tilde{Y}^{(n)}_u$. Boundedness $\Rightarrow$ $[\![\tilde{Y}^{(n)} - \tilde{Y}^{(n,k)}]\!] \xrightarrow[k \to \infty]{} 0$

Suffices to show: $\tilde{Y}^{(n,k)} \in \mathcal{V}_0$.

This follows from:

$$\tilde{Y}^{(n,k)}_u = Z_0 \mathbb{1}_{\{0\}}(u) + \sum_{i=1}^{m_n} \sum_{j=1}^{r_{n,k}} \underbrace{Z^{(n)}_i \mathbb{1}_{\{T_k(t^n_{i-1}) \le j 2^{-k} \le T_k(t^n_i)\}}}_{\in \mathcal{F}_{2^{-k}(j-1)}\text{-meas.}} \mathbb{1}_{(2^{-k}(j-1), 2^{-k} j]}(u)$$

$\boxtimes$