Itô integral wrt. continuous local martingales

last time: M = cont. loc. martingale $\longmapsto \langle M \rangle$ cont., non-decreasing, $\langle M \rangle_0 = 0$
s.t. $\{ M_t^2 - \langle M \rangle_t : t \geq 0 \}$ cont. loc. martingale

Ex $M_t = B_t$, $\langle B \rangle_t = t$
$M_t = B_t^2 - t = 2\int_0^t B_s \, dB_s$, $\langle B^2 - id \rangle_t = 4 \int_0^t B_s^2 \, ds$
$M_t = B_{\phi(t)}$, $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ cont, $\uparrow$, $\langle B_\phi \rangle_t = \phi(t)$

GOAL: Define $\int_0^t Y_s \, dM_s$ for "good" class of Y's.

Recall A process $\{Y_s : s \geq 0\}$ is simple if
$\exists n \in \mathbb{N}$ $\exists\, 0 = t_0 < t_1 < \dots < t_n$ $\exists\, Z_0 \dots Z_n \in L^\infty$:

- $\forall i = 0 \dots n$: Z_i is $\mathcal{F}_{t_{i-1} \vee 0}$-measurable
- $\forall t \geq 0$: $Y_t := 1_{\{0\}}(t) Z_0 + \sum_{i=1}^n Z_i 1_{(t_{i-1}, t_i]}(t)$

$\mathcal{V}_0 := \{ Y : \text{simple} \}$

For $Y \in \mathcal{V}_0$ with above representation:

$$\int_0^t Y_s \, dM_s := \sum_{i=1}^n Z_i \left(M_{t \wedge t_i} - M_{t \wedge t_{i-1}} \right) \qquad \left(\text{RS - approximation} \right)$$

Lemma $\int_0^t Y_s \, dM_s$ does not depend on representation of Y.

Pf As for Riemann integral.

Lemma Let $Y \in \mathcal{V}_0$, and suppose M is continuous L^2-martingale.

Then $\left\{ \int_0^t Y_s \, dM_s : t \geq 0 \right\}$ is continuous L^2-martingale

and Itô "isometry" holds:

$$E\left[\left(\int_0^t Y_s \, dM_s\right)^2\right] = E\left(\int_0^t Y_s^2 \, d\langle M \rangle_s\right)$$

(Stieltjes integral)

Pf L^2 & continuity from setup.
For Itô formula:

$$\text{LHS} = \sum_{i,j=1}^{n} E\left(Z_i Z_j (M_{t\wedge t_i} - M_{t\wedge t_{i-1}})(M_{t_j \wedge t} - M_{t_{j-1}\wedge t}) \right)$$

$$= \sum_{i=1}^{n} E\left(Z_i^2 (M_{t\wedge t_i} - M_{t\wedge t_{i-1}})^2 \right)$$

$$= \sum_{i=1}^{n} E\left(Z_i^2 \left(\langle M\rangle_{t\wedge t_i} - \langle M\rangle_{t\wedge t_{i-1}} \right) \right) = \text{RHS} \quad \boxtimes$$

Notation: $\mathcal{M}_2^{\text{cont}} := \{ M : \text{continuous } L^2\text{-martingale} \}$.

Pseudometric: $$[\![Y - \tilde{Y}]\!]_M := \sum_{n=1}^{\infty} 2^{-n} \min\left\{ 1, \left[E \int_0^n |Y_s - \tilde{Y}_s|^2 \, d\langle M\rangle_s \right]^{1/2} \right\}$$

Denote $$\overline{V_0}^{[\![\cdot]\!]_M} := \left\{ Y : \text{adapted, jointly measurable}, \ \exists \{Y_n\}_{n\in\mathbb{N}} \in V_0^{\mathbb{N}} : [\![Y^{(n)} - Y]\!]_M \to 0 \right\}$$

Lemma Let $M \in \mathcal{M}_2^{cont}$, $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}$. Then $\forall t \geq 0$: $E\left[\int_0^t Y_s^2 \, d\langle M\rangle_s\right] < \infty$.

Moreover, $\forall t \geq 0$ $\exists \int_0^t Y_s \, dM_s$ s.t.

$\forall \{Y^{(n)}\} \in \mathcal{V}_0^{\mathbb{N}}$: $[\![Y^{(n)} - Y]\!]_M \to 0 \Rightarrow \int_0^t Y_s^{(n)} \, dM_s \xrightarrow[n \to \infty]{L^2} \int_0^t Y_s \, dM_s$

In particular, $\int_0^t Y_s \, dM_s \in L^2$ and

$$E\left(\left(\int_0^t Y_s \, dM_s\right)^2\right) = E\left(\int_0^t Y_s^2 \, d\langle M\rangle_s\right)$$

Pf Construction: Let $\{Y^{(n)}\} \in \mathcal{V}_0^{\mathbb{N}}$ be s.t. $[\![Y^{(n)} - Y]\!]_M \to 0$.

Then Itô isometry:

$$E\left(\left|\int_0^t Y_s^{(n)} \, dM_s - \int_0^t Y_s^{(m)} \, dM_s\right|^2\right) = E\left(\left(\int_0^t (Y^{(n)} - Y^{(m)})_s \, dM_s\right)^2\right)$$

$$= E\left(\int_0^t |Y_s^{(n)} - Y_s^{(m)}|^2 \, d\langle M\rangle_s\right) \xrightarrow[n,m \to \infty]{} 0$$

So $\left\{\int_0^t Y^{(n)} \, dM_s\right\}_{n \in \mathbb{N}}$ converges in L^2. Denote the limit $\int_0^t Y_s \, dM_s$. $\boxtimes$

The limit does not depend on $\{Y^{(n)}\}_{n \in \mathbb{N}}$.

Assuming $\mathcal{F}_0$ contains all P-null sets.

Lemma $\forall M \in \mathcal{M}_2^{\text{cont}}$, $\forall Y \in \overline{\mathcal{V}_0}^{[\cdot]_M}$: $\exists \{I_t(Y) : t \geq 0\} \in \mathcal{M}_2^{\text{cont}}$

s.t. $\forall t \geq 0$: $\int_0^t Y_s \, dM_s = I_t(Y)$ a.s.

and $\langle I(Y) \rangle_t = \int_0^t Y_s^2 \, d\langle M \rangle_s$.

Moreover, $\{I_t(Y) : t \geq 0\}$ is unique up to indistinguishability.

Pf. Check: $Y \in \mathcal{V}_0 \Rightarrow \{\int_0^t Y_s \, dM_s : t \geq 0\} \in \mathcal{M}_2^{\text{cont}}$.

Doob L^2:

$$E\left(\sup_{u \leq t} \left| \int_0^u Y_s^{(n)} dM_s - \int_0^u Y_s^{(m)} dM_s \right|^2 \right)$$
$$\leq 4 E\left(\left| \int_0^t Y_s^{(n)} dM_s - \int_0^t Y_s^{(m)} dM_s \right|^2 \right) \xrightarrow[m,n \to \infty]{([Y^{(n)} - Y] \to 0)} 0$$

WLOG; $[Y - Y^{(n)}] \leq 4^{-n}$

Chebyshev: $\sum_{n=1} P\left(\sup_{s \leq t} [\ldots]^2 > 2^{-n} \right) < \infty$.

This defines $I_t(Y)$, continuously, except for null set

L^2-convergence of martingales $\Rightarrow$ limit is a martingale.

Note For $Y \in \mathcal{V}_0$: $\left\langle \int_0^{\cdot} Y_s \, dM_s \right\rangle_t = \int_0^t Y_s^2 \, d\langle M \rangle_s$.

Let $0 \le s \le t$, $A \in \mathcal{F}_s$. Then

$$\begin{aligned} E\left(1_A \, |I_t(Y) - I_s(Y)|^2\right) &= \lim_{n\to\infty} E\left(1_A \left(\int_0^t Y_s^{(n)} dM_s - \int_0^s Y_s^{(n)} dM_s\right)^2\right) \\ &= \lim_{n\to\infty} E\left(1_A \int_s^t \left(Y_s^{(n)}\right)^2 d\langle M \rangle_s\right) \\ &= E\left(1_A \int_s^t Y_s^2 \, d\langle M \rangle_s\right) \end{aligned}$$

$\boxtimes$

Q: What is $\overline{\mathcal{V}_0}^{[\cdot]_M}$?

Prop Suppose $t \mapsto \langle M \rangle_t$ is AC a.s. Then

$$\overline{\mathcal{V}_0}^{[\cdot]_M} = \mathcal{V}_M := \left\{ Y : \text{adapted, jointly meas, } \forall t \ge 0 : E \int_0^t Y_s^2 \, d\langle M \rangle_s < \infty \right\}$$

Prop $\overline{\mathcal{V}_0}^{[\cdot]_M} \supseteq \left\{ Y : \text{adapted, progressively measurable, } \text{———} \,\, '' \,\, \text{————} \right\}$