Rough paths

Motivation: First smooth paths.

Setup: V = normed vector space

$X : [0,T] \longrightarrow V$ conti. function.

X = smooth $:= V^1(X, [0,T]) < \infty$.

Then $\forall\ 0 \le s \le t \le T,\ \forall n \ge 1$:

$$X^n_{s,t} := \int_{s<u_1<\cdots<u_n<t} dX_{u_1} \otimes \cdots \otimes dX_{u_n} \in V^{\otimes n}$$

For $\|\cdot\|$ = norm induced on $V^{\otimes n}$

$$\| X^n_{s,t} \| \le \frac{1}{n!} V^1(X, [0,T])^n$$

Thm (Chen's identity) For X smooth:

$\forall s \le u \le t \ \forall n \ge 1$: $$X^n_{s,t} = \sum_{k=0}^{n} X^k_{s,u} \otimes X^{n-k}_{u,t}$$

where $X^0_{s,t} := 1$.

Pf: $$X^n_{s,t} = \sum_{k=0}^{n} \int_{s<u_1<\cdots<u_k<u<u_{k+1}<\cdots<u_n<t} dX_{u_1} \otimes \cdots \otimes dX_{u_n}$$

$$= \sum_{k=0}^{n} \left(\int_{s<u_1<\cdots<u_k<u} dX_{u_1} \otimes \cdots \otimes dX_{u_k} \right) \otimes \left(\int_{u<u_{k+1}<\cdots<u_n<t} dX_{u_{k+1}} \otimes \cdots \otimes dX_{u_n} \right)$$

This leads to an algebraic structure:

$$S(X) := (1, X^1_{s,t}, X^2_{s,t}, \ldots) \in \bigoplus_{k=0}^{\infty} V^{\otimes k} \quad \ldots \text{ signature of } X.$$

Then Chen's identity becomes: $S(X * Y) = S(X) \otimes S(Y)$

concatenation

Gives rise to Lie group structure

Now axiomatic approach to rough signals

Denote $\Delta_T := \{ (s,t) \in [0,T] \times [0,T] : s \le t \}$

Def Given $n \in \mathbb{N} \setminus \{0\}$, a (continuous) function $X : \Delta_T \to \bigoplus_{k=0}^{n} V^{\otimes k}$ is a multiplicative functional of degree n if $X_{s,t} = (X^0_{s,t}, X^1_{s,t}, \dots, X^n_{s,t})$ obeys:

$$\forall s,t \in \Delta_T : \quad X^0_{s,t} = 1$$

and

$$\forall k = 1 \dots n \;\; \forall 0 \le s \le u \le t \le T : \quad X^k_{s,t} = \sum_{j=0}^{k} X^j_{s,u} \otimes X^{k-j}_{u,t}$$

Ex Iterated integrals of 1-variation function obey this

Ex Same even for p-variation functions with $p<2$

Ex $(1, 0, e \otimes e\,(t-s))$

Ex Itô + Stratonovich examples later

To impose regularity restriction, we need:

Def A <u>control</u> is a map $\omega : \Delta_T \to \mathbb{R}_+$ s.t.

- ω is continous
- $\forall t \in [0,T]: \ \omega(t,t) = 0$
- $\forall 0 \le s \le u \le t \le T: \quad \omega(s,u) + \omega(u,t) \le \omega(s,t)$

<u>Ex</u> $\omega(s,t) = V^p(f, [s,t])$

So "metric" is really $\omega(s,t)^{1/p}$.

<u>Ex</u> $\omega(s,t) = |t-s|^p$

Def: Let $p \ge 1$ be real, $n \ge 1$ an integer, ω a control on $[0,T]$.
A functional $X : \Delta_T \longrightarrow \bigoplus_{k=0}^{n} V^{\otimes k}$ has <u>finite p-variation controlled by ω</u>

if $\forall k = 1 \dots n \ \ \forall s,t \in \Delta_T: \ \|X^k_{st}\| \le \dfrac{\omega(s,t)^{k/p}}{\beta\, \Gamma(k/p + 1)}$

where $\beta \ge \beta_p := 2p^2 \left(1 + \sum_{r \ge 3} \left(\frac{2}{r-2}\right)^{\frac{\lfloor p \rfloor + 1}{2}}\right)$

Ex Itô Brownian path (degree 2)

$$X = \left(1,\ B_t - B_s,\ \tfrac{1}{2}(B_t - B_s)^2 - \tfrac{1}{2}(t-s)\right)$$

Ex Stratonovich Brownian path (degree 2)

$$X = \left(1,\ B_t - B_s,\ \tfrac{1}{2}(B_t - B_s)^2\right)$$

finite p-variation for $p > 2$.

Thm (Extension Thm) Let $p \geq 1$ be real, $n \geq \lfloor p \rfloor$ an integer. Then for each multiplicative functional $X : \Delta_T \to \bigoplus_{k=0}^{n} V^{\otimes k}$ of degree n and finite p-variation w.r.t. some control ω there exists an extension $\bar{X} : \Delta_T \to \bigoplus_{k=0}^{\infty} V^{\otimes k}$ that is multiplicative to any degree and obeys:

$(\beta \geq \beta_p)$

$$\forall k \geq 1\ \forall (s,t) \in \Delta_T : \quad \|\bar{X}^k_{s,t}\| \leq \frac{1}{\beta} \frac{\omega(s,t)^{k/p}}{\Gamma(k/p + 1)}$$

Pf of uniqueness: Let $\bar{X}, \bar{Y}$ be multiplicative of degree $n+1$ extending X. and controlled by ω in p-variation.

Chen's relation: $\forall\ s \le u \le t$:

$$\bar{X}^{n+1}_{s,t} = \bar{X}^{n+1}_{s,u} + \sum_{k=1}^{n} X^{k}_{s,u} X^{n-k+1}_{u,t} + \bar{X}^{n+1}_{u,t}$$

$$\bar{Y}^{n+1}_{s,t} = \bar{Y}^{n+1}_{s,u} + \text{———} \,\text{''}\, \text{———} + \bar{Y}^{n+1}_{u,t}$$

Setting $\psi_{u,v} = \bar{X}^{n+1}_{u,v} - \bar{Y}^{n+1}_{u,v}$ we get

$$\psi_{s,t} = \psi_{s,u} + \psi_{u,t}$$

Setting $\varphi_t := \psi_{0,t}$ we get $\psi_{s,t} = \varphi_t - \varphi_s$.

Now p-variation restriction gives

$$\|\varphi_t - \varphi_s\| = \|\bar{X}^{n+1}_{s,t} - \bar{Y}^{n+1}_{s,t}\| \le K\, \omega(s,t)^{\frac{n+1}{p}}$$

Partition $[0,T]$ into intervals $[t_{i-1},t_i]$ s.t. $\omega(t_{i-1},t_i) < \varepsilon$. (Exists by Cousin's thm).

Then $\sum_{i=1}^{n} \|\varphi_{t_i} - \varphi_{t_{i-1}}\| \le K \sum_{i=1}^{n} \omega(t_{i-1},t_i)^{\frac{n+1}{p}} \le K\varepsilon^{\frac{n+1}{p}-1} \sum_{i=1}^{n} \omega(t_{i-1},t_i) \le K\varepsilon^{\frac{n+1}{p}-1}\omega(0,T)$

So $V^1(\varphi,[0,T]) = 0$ (because $\frac{n+1}{p} > 1$) and so $\bar{X}^{n+1}_{s,t} = \bar{Y}^{n+1}_{s,t}$ $\forall (s,t) \in \Delta_T$. $\boxtimes$

Why does this lead to reasonable theory of ODEs?

Consider ODE: $dY_t = h(Y_t)\,dX_t$ where $X_t : [0,T] \to V$.

Assume h has as many derivatives as you need: Assume X smooth too for now.

$$Y_t = Y_0 + \int_0^t h(Y_s)\,dX_s \qquad (h : W \to L(V,W))$$

$$= Y_0 + h(Y_0)\int_0^t dX_s + \int_0^t (h(Y_s)-h(Y_0))\,dX_s$$

$$= \_\_\_\_ \;\;''\;\; \_\_\_\_ + \int_0^t \left(\int_0^s h'(Y_u)h(Y_u)\,dX_u\right) dX_s$$

<u>Idea</u> Define $h_1(y) = h'(y)h(y)$ & iterate

This gives:

$$Y_t = Y_0 + \sum_{k=1}^{n} h_k(Y_0)\int_{0<u_1<\dots<u_n<t} dX_{u_1}\dots dX_{u_n}$$

$$+ \int_{0<u_1\dots<u_n<t} [h_n(Y_{u_1}) - h_n(Y_0)]\,dX_{u_1}\dots dX_{u_n}$$

where h_k recursively defined by $h_{k+1} = (h_k)'h$