Proof of Picard's thm:

Lemma Let $h \in C^1(\mathbb{R})$ be s.t. $\forall z, \tilde{z} \in \mathbb{R}: \ |h'(z) - h'(\tilde{z})| \le K |z - \tilde{z}|^\alpha$
and h' is bounded. Then

$$\forall y, \tilde{y} \in \mathcal{V}^p(I): \quad \| h\circ y - h\circ \tilde{y} \|_{\mathcal{V}^{p/\alpha}(I)} \le \left(2\|h'\| + 2K\left(\|y\|_{p,I}^\alpha + \|\tilde{y}\|_{p,I}^\alpha \right) \right) \|y - \tilde{y}\|_{\mathcal{V}^p(I)}$$

Note bounds: (Recall: $F(y)(t) := y_0 + \int_0^t h\circ y \, dx$; $y \in \mathcal{V}^p([0,t])$)
$(p < 1+\alpha)$

$$\|F(y)\|_{p,J_T} \le \|h\circ y\|_{\infty,J_T} \|x\|_{p,J_T} + C_{p,p} \|h\circ y\|_{p,J_T} \|x\|_{p,J_T}$$

$$\le \left(|h(y_0)| + 2C_{p,p} \|h'\| \|y\|_p \right) \|x\|_{p,J_T}$$

$$\|F(y) - F(\tilde{y})\|_{p,J_T} \le C_{p,p/\alpha} \|h\circ y - h\circ\tilde{y}\|_{\mathcal{V}^{p/\alpha}(J_T)} \|x\|_{p,J_T}$$

$$\le C_{p,p/\alpha} \left(2\|h'\| + 2K\left(\|y\|^\alpha_{p,J_T} + \|\tilde{y}\|^\alpha_{p,J_T} \right) \right) \|y - \tilde{y}\|_{\mathcal{V}^p(I)} \|x\|_{p,I}$$

$$\|F(y) - F(\tilde{y})\|_{\infty,J_T} \le \text{—} \| \text{—}$$

Assume $r>0$, $T>0$ are s.t.

$$\left(|h(y_0)| + 2C_{p,p}\|h'\| r\right)\|x\|_{p,J_T} \le r$$

$$2C_{p,p/\alpha}\left(2\|h'\| + 4Kr^{\alpha}\right)\|x\|_{p,J_T} \le \frac{1}{2}$$

Then F maps

$$\mathcal{K}_T = \{ y \in \mathcal{V}^p(J_T) : y = y_0 \wedge \|y\|_{p,J_T} \le r \}$$

into itself. Moreover, $\mathcal{K}_T$ is closed in $\mathcal{V}^p(J_T)$. and

$$\forall y, \tilde{y} \in \mathcal{K}_T : \quad \|F(y) - F(\tilde{y})\|_{\mathcal{V}^p(J_T)} \le \frac{1}{2}\|y - \tilde{y}\|_{\mathcal{V}^p(J_T)}$$

So $\exists!\, y \in \mathcal{K}_T : y = F(y)$.

Moreover, the solution is continuous in x: Let $\phi(x)$ = unique solution for x.

$$\phi(x) - \phi(\tilde{x}) = \int \left(h\circ\phi(x) - h\circ\phi(\tilde{x})\right)dx + \int h\circ\phi(\tilde{x})\, d(x-\tilde{x}) \qquad \left(\|\tilde{x}\|_{p,J_T} \le \|x\|_{p,J_T}\right)$$

So $\|\phi(x) - \phi(\tilde{x})\|_{\mathcal{V}^p(J_T)} \le \frac{1}{2}\|\phi(x) - \phi(\tilde{x})\|_{\mathcal{V}^p(J_T)} + \|h'\|\,\|\phi\|_{\mathcal{V}^p(J_T)}\|x-\tilde{x}\|_{p,I}$

So $\tilde{x} \to x$ in $\mathcal{V}^p(J_T) \Rightarrow \phi(\tilde{x}) \to \phi(x)$ in $\mathcal{V}^p(J_T)$. $\boxtimes$

Q: What if $p \geq 2$?

Lemma The map $f, g \mapsto \int_a^b f\,dg$ defined on smooth functions canonically, does not extend continuously to $\mathcal{V}^2([a,b]) \times \mathcal{V}^2([a,b])$.

Pf: Let $f_n(t) := \sum_{k=1}^{n} \frac{1}{\sqrt{k}} b^{-\alpha k} \cos(b^k t)$

$g_n(t) := \sum_{k=1}^{n} \frac{1}{\sqrt{k}} b^{-\alpha k} \sin(b^k t)$.

where $b > 1$. $\alpha \in (0,1)$.

Note $f_n(t) \to f(t) = \sum_{k \geq 1} \frac{1}{\sqrt{k}} b^{-k\alpha} \cos(b^k t)$.

claim $f_n \to f$ in $\mathcal{V}^{1/\alpha}([0,T])$.

Pf
$$|(f-f_n)(t) - (f-f_n)(s)| \leq \sum_{k=n+1}^{\infty} \frac{1}{\sqrt{k}} b^{-k\alpha} |\cos(b^k t) - \cos(b^k s)|$$
$$\leq \frac{1}{\sqrt{n+1}} \left(\sum_{0 \leq k \leq \log_b \frac{1}{|t-s|}} b^{-k\alpha} b^k |t-s| + \sum_{k > \log_b \frac{1}{|t-s|}} 2 b^{-k\alpha} \right)$$
$$\leq \frac{1}{\sqrt{n+1}} \left(\frac{1}{1-b^{-(1-\alpha)}} + \frac{2}{1-b^{-\alpha}} \right) |t-s|^{\alpha}$$

So $\|f - f_n\|_{\frac{1}{\alpha},[0,T]} \le \frac{K}{\sqrt{n+1}} T^{1/\alpha}$ so $f_n \to f$ in $\mathcal{V}^{1/\alpha}([0,T])$

Similarly, $g_n \to g$ in $\mathcal{V}^{1/\alpha}([0,T])$ where $g(t) = \sum_{k=1}^{\infty} \frac{b^{-\alpha k}}{\sqrt{k}} \sin(b^k t)$.

Now set $T := 2\pi$, $b \in \mathbb{N} \setminus \{0,1\}$. Then

$$\int_0^{2\pi} f_n \, dg_n = \sum_{k=1}^{n} \frac{b^{k(1-2\alpha)}}{k} \underbrace{\int_0^{2\pi} \cos(b^k t)^2 \, dt}_{\pi} = \pi \sum_{k=1}^{n} \frac{b^{k(1-2\alpha)}}{k} \xrightarrow[n\to\infty]{\alpha=\frac{1}{2}} \infty$$

Ex Itô vs Stratonovich:

$$\int_0^t B_s \, dB_s \overset{\text{Itô}}{=} \frac{1}{2}(B_t^2 - t)$$

$$\int_0^t B_s \circ dB_s \overset{\text{Stratonovich}}{=} \frac{1}{2} B_t^2$$

Recall:

Thm (Wong-Zakai) Let $\{B^{(n)}\}$ be finite 1-st variation s.t.

$$\forall t \ge 0: \ B^n_t \xrightarrow{P} B_t$$

Then $\forall f \in C^1(\mathbb{R})$:

$$\int_0^t f(B_s^{(n)}) \, dB_s^{(n)} \to \int_0^t f(B_s) \circ dB_s$$

$$= \int_0^t f(B_s) \, dB_s + \frac{1}{2} \int f'(B_s) \, dB_s$$

Natural instinct: Include convergence of integrals under approximation into topology.

To deal with multiple functions: tensorize:

$$f_1(\cdot), f_2(\cdot) \longmapsto X(\cdot) = (f_1(\cdot), f_2(\cdot))$$

$$\int_{s<u<t} dX_u = (f_1(t)-f_1(s), f_2(t)-f_2(s))$$

$$\int_{s<u_1<u_2<t} dX_{u_1} \otimes dX_{u_2} = \begin{pmatrix} \int_s^t f_1\, df_1 & \int_s^t f_1\, df_2 \\ \int_s^t f_2\, df_1 & \int_s^t f_2\, df_2 \end{pmatrix}$$

Why iterated integrals? Recall that, for $p \in [1,2)$:

$$\forall f, g \in V^p([s,t]) : \left| \int_s^t (f - f(s))\, dg \right| \le C_{p,p} \|f\|_{p,[s,t]} \|g\|_{p,[s,t]}$$

and so $\| \int_s^{\cdot} (f - f(s))\, dg \|_{V^p([s,t])} \le C_{p,p} \|f\|_{V^p([s,t])} \|g\|_{p,[s,t]}$

Suppose f, g are α-Hölder

then for $p = 1/\alpha$ $\quad \|f\|_{p,I} \le K |I|^{1/p}$, $\|g\|_{p,I} \le K |I|^{1/p}$

$$\Rightarrow \left| \int_I (f - f(\cdot))\, dg \right| \le K^2 |I|^{2/p} \quad \ldots \quad \int_s^t (f - f(s))\, dg =: \int_{s<u_1<u_2<t} df_{u_1}\, dg_{u_2}$$

For Itô:

$$\int_{s<u_1<u_2<t} dB_{u_1}\, dB_{u_2} = \frac{1}{2}\left[(B_t - B_s)^2 - (t-s)\right] \le K_\varepsilon |t-s|^{1-\varepsilon} \quad \forall \varepsilon > 0$$

$$\int_{s<u_1<u_2<t} dB_{u_1} \circ dB_{u_2} = \frac{1}{2}(B_t - B_s)^2 \qquad \text{— '' —}$$

So $3^{rd}, 4^{th}$ ... iterated integrals

$$\int_{s<u_1<u_2<u_3<t} dB_{u_1} \cdots dB_{u_3}, \ldots$$

are defined as Young integrals !

Idea of rough path theory:

- admit that certain number of iterated integrals are subject to a choice.
- the choice is made subject to Hölder/ p-variation bounds that improve for higher order integrals so that only finite number of choices need to be made.
- different orders of iterated integrals are linked by an algebraic relation.