Generalized Peano and Picard thm

Thm (Generalized Peano) Let $h : \mathbb{R} \to \mathbb{R}$ be locally α-Hölder, $y_0 \in \mathbb{R}$, and $x : \mathbb{R}_+ \to \mathbb{R}$ s.t. $\exists p \in [1, 1+\alpha)\ \forall t \geq 0 :\ x \in \mathcal{V}^p([0,t])$.
Then $\exists T > 0\ \exists y \in \mathcal{V}^p([0,T])$:

$$\forall t \in [0,T] :\quad y(t) = y_0 + \int_0^t h\circ y \, dx$$

where integral meaningful because $h \circ y \in \mathcal{V}^{p/\alpha}([0,T])$ and $\frac{1}{p} + \frac{1}{p/\alpha} = \frac{1+\alpha}{p} > 1$.

Pf: Fix $r > 0$, set $B(y_0, r) := \{ z \in \mathbb{R} : |z - y_0| \leq r \}$.
Let $K \geq 0$ s.t. $\forall z, \tilde{z} \in B(y_0, r) :\ |h(z) - h(\tilde{z})| \leq K |z - \tilde{z}|^\alpha$.
$|h(y_0)| \leq K$
Pick $T > 0$ s.t.

$$\left(2K + C_{p,p/\alpha} K r^\alpha\right) \|x\|_{p, J_T} \leq r \quad \text{where } J_T := [0,T].$$

Note Since $\frac{1}{p} + \frac{1}{p/\alpha} > 1$, $F(y)(t) := y_0 + \int_0^t h \circ y \, dx \quad t \in [0,T]$.
is well defined for all $y \in \mathcal{V}^p([0,T])$

Assume $\|y-y_0\|_{\infty,J_T} \le r \wedge \|y\|_{p,J_T} \le r$. Then

$$\|F(y)-y_0\|_{\infty,J_T} = \left\| \int_0^{\cdot} h\circ y \, dx \right\|_{\infty,J_T} \le |h(y_0)|\,\|x\|_{p,J_T} + C_{p,p/\alpha} \underbrace{\|h\circ y\|_{p/\alpha,J_T}}_{\le K\|y\|^{\alpha}_{p,J_T}} \|x\|_{p,J_T}$$

$$\le \left(K + C_{p,p/\alpha} K r^{\alpha}\right)\|x\|_{x,J_T} \le r$$

and also $\forall I \subseteq J$:

$$\|F(y)\|_{p,I} \le \|h(y)\|_{\infty,J_T}\|x\|_{p,I} + C_{p,p/\alpha}\|h\circ y\|_{p/\alpha,I}\|x\|_{p,I}$$

$$\le \left(2K + C_{p,p/\alpha} K r^{\alpha}\right)\|x\|_{p,I} \le \frac{r}{\|x\|_{p,J_T}}\|x\|_{p,I}$$

Hence: F maps

$$K_T = \left\{ y \in \mathcal{V}^p(J_T) : y = y_0 \wedge \|y-y_0\|_{\infty,J_T} \le r \wedge \forall I \subseteq J : \|y\|_{p,I} \le \frac{r}{\|x\|_{p,J_T}}\|x\|_{p,I} \right\}$$

Now pick $p' \in (p, 1+\alpha)$. By Lemma last time:

K_T is precompact in $\mathcal{V}^{p'}(J_T)$; in fact, it is compact.

Claim: F is continuous on K_T w.r.t. $\|\cdot\|_{\mathcal{V}^{p'}(I)}$.

Take $y^n \to y$ in $(K_T, \|\cdot\|_{p',J_T})$. Then $y^n \to y$ uniformly and so $h\circ y^n \to h\circ y$ uniformly.

Then $\|h\circ y^n - h\circ y\|_{p'/\alpha,J_T} \le \|h\circ y^n - h\circ y\|_{\infty,J}^{1-p/p'}\left(\|h\circ y_n\|_{p/\alpha,I} + \|h\circ y\|_{p/\alpha,I}\right) \le 2K\|x\|^{\alpha}_{p,I}\|h\circ y^n - h\circ y\|_{\infty,J_T} \to 0$

So $\|F\circ y^n - F\circ y\|_{\mathcal{V}^{p'}(J_T)} \le C_{p,p'/\alpha}\|h\circ y^n - h\circ y\|_{\mathcal{V}^{p'/\alpha}}\|x\|_{p/\alpha} \longrightarrow 0$.

Schauder: $\exists y \in K_T : F(y) = y$ $\boxtimes$

__Next Goal:__ Get uniqueness & continuity of $y_0, X \longmapsto y$

__Thm__ (Generalized Picard thm)

Let $h : \mathbb{R} \to \mathbb{R}$ be differentiable with h' uniformly α-Hölder.

Let $y_0 \in \mathbb{R}$, $x : \mathbb{R}_+ \to \mathbb{R}$ s.t. $\exists\, p \in [1, 1+\alpha]$ $\forall t \geq 0 :$ $x \in \mathcal{V}^p([0,t])$.

Then $\exists!\, y : \mathbb{R}_+ \to \mathbb{R}$ with $\forall t \geq 0 :$ $y \in \mathcal{V}^p([0,t])$ s.t.

$$\forall t \geq 0 : \quad y(t) = y_0 + \int_0^t h \circ y \, dx.$$

Moreover, the map $y_0, X \longmapsto y$ is continuous (w.r.t. p-variation metric).

__Lemma:__ Let $h : \mathbb{R} \to \mathbb{R}$ obey $\forall z, \tilde{z} \in \mathbb{R} :$ $|h'(z) - h'(\tilde{z})| \leq K|z - \tilde{z}|^{\alpha}$.

Then

$$\forall y, \tilde{y} \in \mathcal{V}^p(I) : \quad \| h \circ y - h \circ \tilde{y} \|_{\mathcal{V}^{p/\alpha}(I)} \leq \left(2\|h'\| + 2K\left(\|y\|_{p,I}^{\alpha} + \|\tilde{y}\|_{p,I}^{\alpha} \right) \right) \|y - \tilde{y}\|_{\mathcal{V}^p(I)}$$

where $\|h'\| = \sup_{z \in \mathbb{R}} |h'(z)|$

Pf $\quad h(z)-h(\tilde{z}) = g(z,\tilde{z})(z-\tilde{z})$ where $g(z,\tilde{z}) = \int_0^1 h'(sz+(1-s)\tilde{z})\,ds$

$$(*) := h(y_t)-h(\tilde{y}_t) - \big(h(y_s)-h(\tilde{y}_s)\big)$$
$$= g(y_t,\tilde{y}_t)\big(y_t-\tilde{y}_t-(y_s-\tilde{y}_s)\big) + \big(g(y_t,\tilde{y}_t)-g(y_s,\tilde{y}_s)\big)(y_s-\tilde{y}_s)$$

Note: $|g(z,\tilde{z})| \le \|h'\|$

$$|g(z,z')-g(\tilde{z},\tilde{z}')|^{p/\alpha} \le \left| K\int_0^1 ds\, \big|s|z-\tilde{z}|+(1-s)|z'-\tilde{z}'|\big|^{\alpha}\right|^{p/\alpha}$$
$$\le K^{p/\alpha}\big(|z-\tilde{z}|^p+|z'-\tilde{z}'|^p\big)$$

Then $$|(*)|^{p/\alpha} \le 2^{p/\alpha}\|h'\|^{p/\alpha}\,|y_t-\tilde{y}_t-(y_s-\tilde{y}_s)|^{p/\alpha} + 2^{p/\alpha}K^{p/\alpha}\big(|y_t-\tilde{y}_t|^p+|y_s-\tilde{y}_s|^p\big)\underbrace{|y_s-\tilde{y}_s|^{p/\alpha}}_{\le \|y-\tilde{y}\|_{\infty,I}^{p/\alpha}}$$

Hence $$V^{p/\alpha}(h\circ y-h\circ\tilde{y},I) \le 2^{p/\alpha}\|h'\|^{p/\alpha}\,V^{p/\alpha}(y-\tilde{y},I)$$
$$+\,2^{p/\alpha}K^{p/\alpha}\big(V^p(y,I)+V^p(\tilde{y},I)\big)\|y-\tilde{y}\|_{\infty,I}^{p/\alpha}$$

So
$$\|h\circ y-h\circ\tilde{y}\|_{p/\alpha} \le 2\|h'\|\,\|y-\tilde{y}\|_{p/\alpha,I} + 2K\big(\|y\|_{p,I}^{\alpha}+\|\tilde{y}\|_{p,I}^{\alpha}\big)\|y-\tilde{y}\|_{\infty,I}$$

Now combine with $\|h\circ y-h\circ\tilde{y}\|_{\infty,I} \le \|h'\|\,\|y-\tilde{y}\|_{\infty,I}$ & use monotonicity of $p\mapsto\|y-\tilde{y}\|_{p,I}$. $\boxtimes$

Pf of Picard Observe the following bounds:

$$\|F \circ y\|_{p,I} \leq \|h \circ y\|_{\infty,I} \|x\|_{p,I} + C_{p,p} \|h \circ y\|_{p,I} \|x\|_{p,I}$$

$$\|h \circ y\|_{\infty,I} \leq |h(y_0)| + \|h'\| \underbrace{\|y - y_0\|_{\infty,I}}_{\leq \|y\|_{p,I}}$$

$$\|h \circ y\|_{p,I} \leq \|h'\| \|y\|_{p,I}$$

Take $M \geq 2|h(y_0)|$ and find $T > 0$ s.t.

$$\|x\|_{p,J_T} \leq 1$$

$$(1 + C_{p,p}) \|x\|_{p,J_T} \leq \frac{1}{2}$$

$$\left(2\|h'\| + 4KM^{\alpha}\right) \|x\|_{p,J_T} \leq \frac{1}{2}$$

Then $\|F \circ y\|_{p,J_T} \leq \left(|h(y_0)| + \frac{1}{2}\|y\|_{p,I}\right) \|x\|_{p,I}$