Solving moderately rough ODEs

recall $V^p(f, I) = \sup_{\pi} V^p(f, I, \pi)$

$\mathcal{V}^p(I) := \{ f \in C(I) : V^p(f, I) < \infty \}$, $\quad \|f\|_{p,I} := V^p(f,I)^{1/p}$,

$(\mathcal{V}^p(I), \|\cdot\|_{\mathcal{V}^p})$ ... Banach space. $\quad \|f\|_{\mathcal{V}^p} := \|f\|_{\infty,I} + \|f\|_{p,I}$.

$\|f\|_{\infty,I}$: sup-norm.

Lemma If $\exists K \geq 0$ s.t. $\forall x, y \in I : |f(x) - f(y)| \leq K|x-y|^\alpha$
then $f \in \mathcal{V}^{1/\alpha}(I)$ and $\|f\|_{1/\alpha, I} \leq K|I|^{1/\alpha}$.

Lemma Suppose $\exists c > 0$ $\exists \alpha \in (0,1]$ and $f : I \to \mathbb{R}$ s.t.

$$\mathcal{J} := \{ [s,t] \subseteq I : (\exists [s',t'] \subseteq [s,t] : |f(t') - f(s')| \geq c|s-t|^\alpha) \}$$

has Cousin property: $\forall x \in I$ $\exists \delta > 0$ $\forall J \subseteq I$: $x \in J \wedge |J| < \delta \Rightarrow J \in \mathcal{J}$.
Then $f \notin \mathcal{V}^p(I)$ for $p < 1/\alpha$.

Pf. Cousin property: $\forall \delta > 0 \ \exists \Pi = \{t_i\}_{i=0}^n$ partition s.t. $\|\Pi\| \le \delta \wedge \forall i = 1..n: [t_{i-1}, t_i] \in \mathcal{J}$.

For each i, let $[s_i', t_i'] \subseteq [t_{i-1}, t_i]$ s.t. $|f(s_i') - f(t_i')| \ge c\,|t_i - t_{i-1}|^\alpha$.

Now let Π' be partition with points $\{t_i, t_i', s_i'\}_{i=1}^n$.

Then
$$V^p(f, I, \Pi') \ge \sum_{i=1}^n |f(s_i') - f(t_i')|^p$$
$$\ge c^p \sum_{i=1}^n |t_i - t_{i-1}|^{\alpha p} \overset{\alpha p < 1}{\ge} c^p \delta^{\alpha p - 1} |I| \xrightarrow[\delta \downarrow 0]{\alpha p < 1} \infty. \ \boxtimes$$

<u>Note</u> For $f \notin V^p(I)$ for $p < 1/\alpha$ it suffices to show that $\inf_{t \in I} \limsup_{s \downarrow t} \dfrac{|f(t) - f(s)|}{|t - s|^\alpha} > 0$.

<u>Lemma</u> (Invariance under reparametrization). Let $\varphi : I \to I$ be continuous, increasing bijection. Then $\forall f \in V^p(I)$: $f \circ \varphi \in V^p(I)$ and, in fact:
$$\|f \circ \varphi\|_{p,I} = \|f\|_{p,I}.$$

Pf: Let Π = partition of I, $\Pi = \{t_i\}_{i=0}^n$. Set $\Pi' = \{\varphi(t_i)\}_{i=0}^n$ ... partition as well.
$$V^p(f \circ \varphi, I, \Pi) = V^p(f, I, \Pi') \le V^p(f, I)$$
This shows $\|f \circ \varphi\|_{p,I} \le \|f\|_{p,I}$. Passing to φ^{-1} gives another inequality. $\boxtimes$

Lemma Suppose $f \in \mathcal{V}^p(I)$ is <u>not</u> constant on any interval.
Then $\exists\, \varphi : I \to I$ continuous, increasing bijection s.t.

$$\forall s,t \in I : \quad |f\circ\varphi(t) - f\circ\varphi(s)| \leq \left(\frac{V^p(f, I)}{|I|}\right)^{1/p} |t-s|^{1/p}$$

Pf take φ to be inverse of $t \longmapsto a + \frac{b-a}{V^p(f,I)} V^p(f,[a,t])$.

Bad feature of invariance under reparametrization:

$\mathcal{V}^p(I)$ is NOT separable.

Above lemma uses:

Lemma Let f be s.t. $V^p(f,[a,b]) < \infty$. Define $v_f : [a,b] \to \mathbb{R}_+$ by $v_f(a) := 0$ and $\forall t \in (a,b] : v_f(t) = V^p(f,[a,t])$.
Then v_f is non-decreasing with

$$\forall a \leq s < t \leq b : \quad V^p(f,[s,t]) \leq v_f(t) - v_f(s)$$

and

$$f \in C[a,b] \Rightarrow v_f \in C[a,b].$$

Lemma Let $h:\mathbb{R}\to\mathbb{R}$ be α-Hölder: $\exists K>0\ \forall x,y\in\mathbb{R}:\ |h(x)-h(y)|\le K|x-y|^{\alpha}$.

Then $\forall p\ge 1\ \forall f\in V^{p}(I)$: $h\circ f\in V^{p/\alpha}(I)$ and $\|h\circ f\|_{p/\alpha,I}\le K\|f\|_{p}^{\alpha}$.

Pf $V(h\circ f, I, \Pi)=\sum_{i=1}^{n}|h\circ f(t_i)-h\circ f(t_{i-1})|^{p}\le K^{p}\sum_{i=1}^{n}|f(t_i)-f(t_{i-1})|^{\alpha p}$ $\boxtimes$

Now towards ODE:

$$dy_t = h(y_t,t)\,dx_t + g(y_t,t)\,dt.$$

here x,y are vector space valued,
so 1-dimensional trix are out !!
... reduce to $dy_t=h(y_t)\,dx_t$.

Thm (Peano theorem) Let $h\in C(\mathbb{R})$, $y_0\in\mathbb{R}$, $x\in V^{1}([0,t])$ for all $t>0$.
Then $\exists T>0\ \exists y\in V^{1}([0,T])\cap C([0,T])$:

$$\forall t\in[0,T]:\ y(t)=y_0+\int_0^t h(y)\,dx$$

Stieltjes integral.

Pf Fix $r>0$. Let $K := \sup\{|h(z)| : |z-y_0| \le r\}$. WLOG $K>0$.
Let $T>0$ be s.t. $V'(x,[0,T]) \le r/K$. (continuity of V').
Denot $J_T := [0,T]$.
Note: For $y \in C(J_T)$: $F(y)(t) := y_0 + \int_0^t h\circ y \, dx$ is well def.
GOAL: Find a fixed point of F.

Note If $\|y-y_0\|_{\infty,J_T} \le r$ then:
$$\|F(y)-y_0\|_{\infty,J_T} \le K\,V'(x,J_T) \le K\frac{r}{K} = r.$$

and also: $\forall I \subseteq J_T$: $V'(F(y),I) \le K\,V'(x,I)$.

It follows that F maps
$$\mathcal{K}_T := \{ y \in V'(J_T) : y(0)=y_0 \wedge \|y-y_0\|_{\infty,J_T} \le r \wedge (\ast) \}.$$
into itself.
Also: $\mathcal{K}_T$ is bdd, closed in $C(J_T)$ and equicontinuous
so by Arzelà-Ascoli: $\mathcal{K}_T$ is compact in $C(J_T)$.

Claim $y \mapsto F(y)$ is continuous in $\|\cdot\|_{\infty, J_T}$.

Pf. if $\|y^{(n)} - y\|_{\infty, J_T} \to 0$ then $\|h \circ y_n - h \circ y\|_{\infty, J_T} \longrightarrow 0$

The $\|F \circ y^n - F \circ y\|_{\infty, J_T} \leq \|h \circ y_n - h \circ y\|_{\infty, J_T} V^1(x, J_T) \longrightarrow 0.$

So F is a continuous function mapping compact set in a Banach space into itself. By Schauder's fixed pt. thm: $\exists y \in K_T : F(y) = y$. $\boxed{X}$

Thm (Generalized Peano) Suppose h is locally α-Hölder, $y_0 \in \mathbb{R}$, $x \in V^p([0,t])$ for all $t > 0$ and some $p < 1 + \alpha$:

Then $\exists T > 0$ $\exists y \in V^p([0,T]) \cap C([0,T])$:

$$\forall t \in [0,T] : \quad y(t) = y_0 + \int_0^t h(y)\, dx$$

Stieltjes integral.

Lemma: Let $p \geq 1$ and suppose $\mathcal{K} \subseteq \mathcal{V}_0^p(I)$ is bounded in $\|\cdot\|_{p,I}$-norm and is uniformly equicontinuous. Then

$$\forall q > p: \quad \mathcal{K} \text{ is precompact in } \mathcal{V}^q(I).$$

Pf: Let $\{y^{(n)}\}_{n\in\mathbb{N}} \in \mathcal{K}^{\mathbb{N}}$. Since $\mathcal{K}$ is bounded in $C(I)$, Arzela-Ascoli given us $\{y^{(n_k)}\}_{k\in\mathbb{N}}$ converging uniformly. Now oscillation ineq:

$$\|y^{(n)} - y^{(m)}\|_{q,I} \leq \operatorname{osc}(y^{(n)} - y^{(m)}, I)^{1-p/q} \, \underbrace{\|y^{(n)} - y^{(m)}\|_{p,I}^{p/q}}_{\text{bounded by assumption}}$$