Continuing the proof of Young criterion

$p' > p,\ q' > q,\ \frac{p'}{p} = \frac{q'}{q}$

$\frac{1}{p'} + \frac{1}{q'} > 1$

$$|S(f,g,\Pi) - S(f,g,\Pi')|$$

$$\leq C_{p,q} \underbrace{\left[\max_{i=1\dots n} \operatorname{osc}(f,[t_{i-1},t_i]) \operatorname{osc}(g,[t_{i-1},t_i]) \right]^{1-p/p'}}_{O(f,g,\Pi)} \underbrace{\left[\sum_{i=1}^n V^p(f,[t_{i-1},t_i])^{1/p'} \right]}_{\leq \left[\sum_{i=1}^n V^p(f,[t_{i-1},t_i])\right]^{1/p'}} \left[\sum_{i=1}^n V^q(g,[t_{i-1},t_i])^{1/q'} \right]$$

$$\leq \left[\sum_{i=1}^n V^p(f,[t_{i-1},t_i])\right]^{1/p'} \leq V^p(f,[a,b])^{1/p'}$$

Lemma For f, g bounded with no common discontinuity points

$$\lim_{\|\Pi\| \to 0} O(f,g,\Pi) = 0$$

Pf: By contradiction: $O(f,g,\Pi_n) \geq 0$ for $\|\Pi_n\| \to 0$ implies, by compactness

$\exists \varepsilon > 0\ \exists x \in [a,b]\ \forall \delta > 0 : \operatorname{Osc}(f,[x-\delta,x+\delta]) \operatorname{Osc}(g,[x-\delta,x+\delta]) \geq \varepsilon$

$\Rightarrow$ contradicts assumption that at least one of f and g is continuous at x. $\boxtimes$

Hence we get $\lim_{\|\Pi\|, \|\Pi'\| \to 0} |S(f, g, \Pi) - S(f, g, \Pi')| = 0$

which implies RS-integrability. $\boxtimes$

Corollary: $\left| \int_a^b (f - f(a))\, dg \right| \le C_{p,q} V^q(f)^{1/q} V^p(g)^{1/p}$ (whenever RHS $< \infty$ & $1/p + 1/q > 1$)

Pf Follows from Love-Young inequality by taking limits.

... $\left| \int_a^b f\, dg \right| \le \|f\|_{C[a,b]} \|g\|_{BV[a,b]}$

Corollary: $\left| \int_a^b f\, dg - \sum_{i=1}^n f(t_i^*)[g(t_i) - g(t_{i-1})] \right|$

$$\le C_{p,q} \left(\sum_{i=1}^n V^p(f, [t_{i-1}, t_i])^{1/p} \right) \left(\sum_{i=1}^n V^q(g, [t_{i-1}, t_i])^{1/q} \right)$$

Solving ODE's driven by moderately rough paths

GOAL ODE $dy_t = h(y_t)dx_t$ ... $y_t = y_0 + \int_0^t h(y_s)dx_s$
will x haves only p-variation for some $p > 1$..
(with product so general that explicit solution are out).

Will need to restrict to $p < 2$

Need: Theory of function of bounded p-variation.

Def: $\| f \|_{p, I} := V^p(f, I)^{1/p} \quad (p \geq 1)$

Lemma $f \mapsto \| f \|$ is homogeneous, subadditive, non-negative.
vanishing only if f is constant. (for $p \geq 1$)

Def $\mathcal{V}^p[a,b] := \{ f \in C[a,b] : V^p(f,[a,b]) < \infty \}$.

$\mathcal{V}^p_0[a,b] := \{ f \in \mathcal{V}^p[a,b] : f(a) = 0 \}$.

$\|f\|_{\mathcal{V}^p[a,b]} := \sup_{x \in [a,b]} |f(x)| + V^p(f,[a,b])^{1/p}$.

Lemma $\|\cdot\|_{\mathcal{V}^p(I)}$ is a norm on $\mathcal{V}^p(I)$ $\big($ for $p \in [1,\infty)$ $\big)$

$\|\cdot\|_{p,I}$ — " — $\mathcal{V}^p_0(I)$.

Lemma: $\forall f \in C(I)$:

- $\forall\, 1 \le p < q$: $\|f\|_{p,I} \ge \|f\|_{q,I}$
- $p \longmapsto \log \|f\|_{p,I}^p$ convex & continuous where finite.
- $\mathrm{osc}(f,I) \le \|f\|_{p,I}$ $\Big(\lim_{p\to\infty} \|f\|_{p,I} = \mathrm{osc}(f,I)$ where $\lim^t$ is finite $\Big)$
- $\forall p < q$: $\|f\|_{q,I} \le \mathrm{osc}(f,A)^{1-p/q} \, \|f\|_{p,I}^{p/q}$

Lemma $\forall f \in C(I)\ \forall \{f_n\}_{n\in\mathbb{N}} \in C(I)^{\mathbb{N}}$: $f_n \to f$ pointwise implies

$$\|f\|_{p,I} \le \liminf_{n\to\infty} \|f_n\|_{p,I}$$

Pf $\|f\|_{p,I} < \infty \Rightarrow \forall \varepsilon\ \exists \Pi\ :\ V^p(f,I,\Pi) \ge V^p(f,I) - \varepsilon$.

Then $\liminf V^p(f_n,I) \ge \liminf_{n\to\infty} V^p(f_n,I,\Pi) = V^p(f,I,\Pi) \ge$

$\|f\| = \infty \Rightarrow \forall M>0\ \exists \Pi : V^p(f,I,\Pi) \ge M$. Same arg: $\liminf_{n\to\infty} V^p(f_n,I) \ge M$. $\boxtimes$

Lemma $\forall p \in [1,\infty)$: $(\mathcal{V}^p(I), \|\cdot\|_{\mathcal{V}^p(I)})$, $(\mathcal{V}^p_0(I), \|\cdot\|_{p,I})$ are Banach spaces.

Pf: NTS completeness. so $\|f_n - f_m\|_{p,I} \to 0 \Rightarrow \exists f : \|f_n - f\| \to 0$

oscillation bound: $\|f_n - f_m\|_{p,I} \to 0 \Rightarrow f(t) := \lim_{n\to\infty} f_n(t)$ exists,

then Lemma: $\|f_m - f\|_{p,I} \le \liminf_{n\to\infty} \|f_n - f_m\| \xrightarrow[m\to\infty]{} 0$. $\boxtimes$

Cor. $\forall\, 1 \le p \le q$: $\mathcal{V}_0^1(I) \subseteq \mathcal{V}^p(I) \subseteq \mathcal{V}^q(I) \subseteq C_0(I)$

with continuous embeddings.

Lemma If f is α-Hölder in the sense $\forall s,t \in I$: $|f(t)-f(s)| \le K|t-s|^\alpha$ and f vanishes at left endpoint, then

$$f \in \mathcal{V}_0^{1/\alpha}(I) \wedge \|f\|_{1/\alpha, I} \le K.$$

Ex Weierstrass functions:

$$W_\alpha(t) := \sum_{n=0}^{\infty} b^{-n\alpha} \cos(2\pi b^n t)$$

($b>1$ odd natural) $\alpha \in (0,1)$. Then W_α is α-Hölder.

Ex $\{B_t : t \ge 0\} = \text{SBM}$ is α-Hölder for $\alpha < 1/2$.

$\{B_{B_t} : t \ge 0\}$, $\{B_{B_{B_t}} : t \ge 0\}$, ... $\frac{1}{2^n} - \varepsilon$ Hölder processes

Ex Fractional Brownian Motion:

$\{B_t : t \geq 0\}$ Gaussian process, mean-zero

$$\mathrm{Cov}(B_s, B_t) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right)$$

where H = Hurst index, $H \in (0,1)$. Key pt: $\mathrm{Var}(B_t - B_s) = |t-s|^{2H}$

α-Hölder continuous (version) for all $\alpha < H$.

$H = 1/2 \Rightarrow$ SBM.