Young integral (L.C. Young)

GOAL: Solve ODE $dy_t = f(y_t)\,dx_t$ for "rough" noise process x_t

Need improved theory of Stieltjes integral $\int_a^b f\,dg$

Reminders: $f, g : [a,b] \to \mathbb{R}$, $\Pi = (\{t_i\}_{i=0}^n, \{t_i^*\}_{i=1}^n)$ marked partition of $[a,b]$

($t_0 = a < t_1 < \dots < t_n = b$, $\forall i = 1 \dots n$: $t_i^* \in [t_{i-1}, t_i]$).

Let $S(f,g,\Pi) := \sum_{i=1}^n f(t_i^*)\left[g(t_i) - g(t_{i-1})\right]$ Riemann-Stieltjes sum

Mesh of Π: $\|\Pi\| := \max_{i=1\dots n} |t_i - t_{i-1}|$

Def f is Riemann-Stieltjes integrable w.r.t. g on $[a,b]$ if

$\exists L \in \mathbb{R}\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall \Pi =$ marked partition of $[a,b]$:

$$\|\Pi\| < \delta \Rightarrow \left| S(f,g,\Pi) - L \right| < \varepsilon.$$

Criteria for RS-integrability:

Lemma If $f \in C[a,b]$, $g \in BV[a,b]$ $\Longrightarrow$ f is RS-integrable w.r.t g.

Moreover, $$\left|\int_a^b f\, dg\right| \le \|f\|_{C[a,b]} \underbrace{\|g\|_{BV[a,b]}}_{V^1(g,[a,b])}$$

Lemma $\forall f, g : [a,b] \to \mathbb{R}$. f RS-integrable w.r.t g $\Longleftrightarrow$ g RS-integrable w.r.t f.

Moreover, $$\int_a^b f\, dg = f(b)g(b) - f(a)g(a) - \int_a^b g\, df.$$

LC Young discovered a new criterion that interpolates between these:

$$V^p(f, [a,b], \Pi) := \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p$$

$$V^p(f, [a,b]) = \sup_{\Pi} V^p(f, [a,b], \Pi). \quad \quad p \ge 1$$

We say f is bounded p-variation if $V^p(f,[a,b]) < \infty$.

Thm (LC Young, 1936) Let $f, g : [a,b] \to \mathbb{R}$ be s.t.

$\exists\, p, q \geq 1 : \ \frac{1}{p} + \frac{1}{q} > 1 \ \wedge \ V^p(f, [a,b]) < \infty \ \wedge \ V^q(g, [a,b]) < \infty$,

and f and g have **no** common points of discontinuity. Then f is RS-integrable w.r.t. g on $[a,b]$

Prop (Love-Young inequality)

For any marked partition $(\{t_i\}_{i=0}^n, \{t_i^*\}_{i=1}^n)$, any $t^* \in [a,b]$ and any $p, q > 0$:

$$\left| f(t^*)\left[g(b) - g(a)\right] - \sum_{i=1}^{n} f(t_i^*)\left[g(t_i) - g(t_{i-1})\right] \right| \leq \left[1 + \zeta_n\left(\tfrac{1}{p} + \tfrac{1}{q}\right)\right] V^p(f, [a,b])^{1/p}\, V^q(g, [a,b])^{1/q}$$

where $\zeta_n(s) = \sum_{k=1}^{n-1} k^{-s}$.

Lemma $\forall p,q>0 \ \forall n\geq 1 \ \forall a_1, \dots, a_n, b_1, \dots, b_n \geq 0$:

$$\min_{i=1,\dots,n} a_i b_i \leq \left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^{1/p}\left(\frac{1}{n}\sum_{i=1}^{n} b_i^q\right)^{1/q}$$

Pf.
$$\min_{i=1,\dots,n} a_i b_i \leq \left(\prod_{i=1}^{n} a_i^p\right)^{\frac{1}{np}}\left(\prod_{i=1}^{n} b_i^q\right)^{\frac{1}{nq}} \overset{\text{AMGM}}{\leq} \left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^{1/p}\left(\frac{1}{n}\sum_{i=1}^{n} b_i^q\right)^{1/q}. \qquad \boxtimes$$

This implies
$$\sum_{i=1}^{n} a_i b_i \leq \zeta_{n+1}\left(1/p+1/q\right)\left(\sum_{i=1}^{n} a_i^p\right)^{1/p}\left(\sum_{i=1}^{n} b_i^q\right)^{1/q}$$

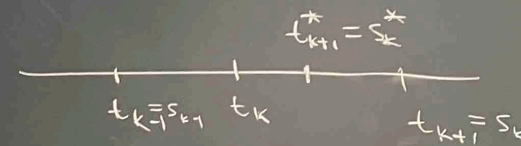
Proof of Proposition: Induction on n:

$\boxed{n=1}$
$$\left| f(t^*)[g(b)-g(a)] - f(s^*)[g(b)-g(a)] \right|$$
$$\leq |f(t^*)-f(s^*)|\,|g(b)-g(a)| \leq V^p(f)^{1/p}\, V^q(g)^{1/q}.$$

Now assume bound holds up to $n-1$. Let $(\{t_i\}_{i=0}^n, \{t_i^*\}_{i=1}^n)$ be marked partition.

Given $k = 1, \dots, n-1$ TBD, let

$$(s_i, s_i^*) := \begin{cases} (t_i, t_i^*) & i < k \\ (t_{i+1}, t_{i+1}^*) & i \geq k \end{cases}$$



Now compute

$$(*) := \sum_{i=1}^{n-1} f(s_i^*)[g(s_i) - g(s_{i-1})] - \sum_{i=1}^{n} f(t_i^*)[g(t_i) - g(t_{i-1})]$$

$$= f(t_{k+1}^*)[g(t_{k+1}) - g(t_{k-1})] - f(t_k^*)[g(t_k) - g(t_{k-1})] - f(t_{k+1}^*)[g(t_{k+1}) - g(t_k)]$$

$$= [f(t_{k+1}^*) - f(t_k^*)][g(t_k) - g(t_{k-1})]$$

Now assume k minimizes absolute value of this term.

Lemma gives: $|(*)| \le (n-1)^{-1/p-1/q} \, V^p(f)^{1/p} \, V^q(g)^{1/q}$.

Induction step completed by Δ-ineq. and $\zeta_n(s) = \zeta_{n-1}(s) + (n-1)^{-s}$. $\boxtimes$

For theorem we need

Lemma $\forall p,q \ge 1 \; \forall a_1 \ldots a_n, b_1 \ldots b_n \ge 0$: $\frac{1}{p} + \frac{1}{q} \ge 1 \Rightarrow \sum_{i=1}^n a_i b_i \le \left(\sum_{i=1}^n a_i^p\right)^{1/p} \left(\sum_{i=1}^n b_i^q\right)^{1/q}$

Pf: Hölder + $\forall \alpha \in (0,1] \; \forall x_1 \ldots x_n \ge 0$: $\sum_{i=1}^n x_i \le \left(\sum_{i=1}^n x_i^\alpha\right)^{1/\alpha}$

Pf of Thm: Let $p' > p$, $q' > q$ be s.t. $\frac{1}{p'} + \frac{1}{q'} > 1 \wedge \frac{p}{p'} = \frac{q}{q'}$.

Prop gives: If Π, Π' = marked partition s.t. each interval of Π is a union of intervals of Π' then

$$\left| S(f,g,\Pi) - S(f,g,\Pi') \right| \le C_{p'q'} \sum_{i=1}^n V^{p'}(f,[t_{i-1},t_i])^{1/p'} V^{q'}(g,[t_{i-1},t_i])^{1/q'}$$

where $\{t_i\}_{i=0}^n$ are partition points of Π and $C_{p'q'} = 1 + \zeta(1/p' + 1/q')$

To be finished tomorrow ...