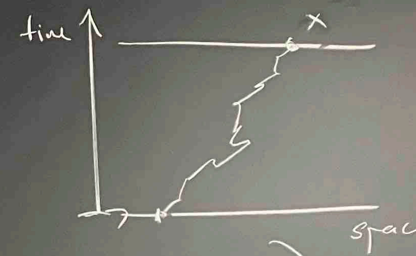
Solving CH-KPZ via FKF

Following Bertini-Cancrini J. Stat. Phys. '95



Last time: $\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + u(t,x) f(t,x)$

solved by $u(t,x) = E^x\left(u_0(B_t) \exp\left\{ \int_0^t f(r, B_{t-r})\, dr \right\} \right)$

Feynman-Kac formula

$$\int_{[0,t]\times\mathbb{R}^d} f(r,y)\,\delta(y-B_{t-r})\,dr\,dy$$

Unique under subgaussian growth of u_0
subquadratic — " — f

Now back to KPZ:

Idea: Approximate solution:

bump-function of width ε (in $\mathbb{R}^d$)

$$t,x \mapsto E^x\left(u_0(B_t) \exp\left\{ \int_{[0,t]\times\mathbb{R}^d} \varphi_\varepsilon(y-B_{t-r})\, W(dr\,dy) \right\} \right)$$

meaningful when

$$\mathrm{Var}(\text{integral}) = \int_0^t dr \int_{\mathbb{R}} dy\, \varphi_\varepsilon(y-B_{t-r})^2 = t \int dy\, \varphi_\varepsilon(y)^2 = t\, \varphi_\varepsilon * \varphi_\varepsilon(0) \sim \varepsilon^{-d}$$

Issue: Absence of Wick ordering?!

Resolved by setting

$$u^\varepsilon(t,x) := E^x\Big(u_0(B_t) : \exp\Big\{ \int_0^t \varphi_\varepsilon(y - B_{t-r})\, W(dr\,dy)\Big\} : \Big)$$

$$= E^x\Big(u_0(B_t) \exp\Big\{ \int_0^t \varphi_\varepsilon(y - B_{t-r})\, W(dt\,dy) - \frac{t}{2}\, \varphi_\varepsilon * \varphi_\varepsilon(0)\Big\}\Big)$$

expansion

$$u^\varepsilon(t,x) = E^x\big(u_0(B_t)\big) + \sum_{n=1}^{\infty} \frac{1}{n!} E^x\Big(u_0(B_t) : \Big(\int_0^t \varphi_\varepsilon(y - B_{t-r})\, W(dr\,dy)\Big)^n : \Big)$$

$$= g_t * u_0(x) + \sum_{n=1}^{\infty} \int_{\substack{0<t_1<\dots<t_n\le t \\ x_1,\dots,x_n\in\mathbb{R}^d}} \prod_{k=2}^{n+1} g_{t_k - t_{k-1}}(x_k - x_{k-1})\, g_{t_1} * u_0(x_1) \prod_{k=1}^{n} \varphi_\varepsilon(y_k - x_k)\, dx_1 \dots dx_n\, W(dt_1\,dy_1) \dots W(dt_n\,dy_n)$$

Lemma Let $d=1$.

$$\forall x\in\mathbb{R}\ \forall t\ge 0:\quad u^\varepsilon(t,x) \xrightarrow[\varepsilon\downarrow 0]{L^2} g_t * u_0(x) + \sum_{n=1}^{\infty} \int_{0<t_1<\dots<t_n\le t} \prod_{k=2}^{n+1} g_{t_k - t_{k-1}}(x_k - x_{k-1})\, g_{t_1} * u_0(x)\, W(dt_1\,dx_1)\dots W(dt_n\,dx_n)$$

In short, u^ε converges to CH-solution obtained earlier.

Focus on n-th term: (the L^2-norm of difference)

We get:

$$\int \Big(\int \prod_{k=2}^{n+1} g_{t_k - t_{k-1}}(x_k - x_{k-1})\, g_{t_1} * u_0(x_1) \prod_{i=1}^{n} \varphi_\varepsilon(\underbrace{y_i - x_i}_{z_i})\, dx_1 \dots dx_n - \prod_{k=2}^{n+1} g_{t_k - t_{k-1}}(y_k - y_{k-1})\, g_{t_1} * u_0(y_1) \Big)^2 dt_1 \dots dt_n\, dy_1 \dots dy_n$$

Jensen

$$\underset{z_i = x_i - y_i}{\le} \int \Big(\prod_{k=2}^{n+1} g_{t_k - t_{k-1}}(y_k - y_{k-1} + z_k - z_{k-1})\, g_{t_1} * u_0(y_1 + z_1) - \prod_{k=2}^{n} \dots \quad (z=0) \quad \dots \Big)^2 \prod_{i=1}^{n} \varphi_\varepsilon(z_i)\, dz_i \prod_{i=1}^{n} dt_i\, dy_i$$

Combining the gaussian kernels & using additive property of Gaussians:

expression bounded by

$$\int_{0 \le t_1 < \dots < t_n \le t} \prod_{i=1}^{n} \frac{dt_i}{\sqrt{4\pi(t_i - t_{i-1})}}\; \underbrace{E\Big(\big[u_0(x+Y+Z) - u_0(x+Y) \big]^2 \Big)}_{\le \frac{1}{\sqrt{2\pi t}} E\left(\| u_0(\cdot + Z) - u_0 \|^2_{L^2(\mathbb{R})} \right)}$$

where $Y = N(0,t)$

$Z = \sum_{i=1}^{n} Z_i$, $Z_i \dots$ iid $\sim \varphi_\varepsilon(z)\, dz$

So the difference of n-th terms has 2^{nd} moment bounded by

$$\le \frac{C^n t^n}{\Gamma(\frac{n}{2}+1)} \int_{\mathbb{R}^n} \underbrace{\| u(\cdot + \sum_{i=1}^{n} z_i \varepsilon) - u(\cdot) \|_2^2}_{\le 4 \|u\|_2^2} \prod_{i=1}^{n} \varphi_1(z_i)\, dz_i \xrightarrow[\varepsilon \downarrow 0]{DCT} 0$$

Q: Does FKF for u^ε have a limit as $\varepsilon \downarrow 0$.

i.e. Can we define

$$E^x\left(u_0(B_t) : \exp\left\{ \int_{[0,t]\times\mathbb{R}} \delta(y - B_{t-r})\, W(dr\,dy) \right\} : \right)$$

Lemma Denote $M_t^\varepsilon := E^x\left(u_0(B_t) \exp\left\{ \int_{[0,t]\times\mathbb{R}} \varphi_\varepsilon(y - B_{t-r}) W(dr\,dy) - \frac{t}{2}\varphi_\varepsilon * \varphi_\varepsilon(0) \right\} \right)$

$$\lim_{\varepsilon,\delta \downarrow 0} \| M_t^\varepsilon - M_t^\delta \|_{L^2(P)} = 0$$

$$\partial_t u^\varepsilon(t,x) = \partial_{xx} u^\varepsilon(t,x) + u(t,x) W(t,x) - \frac{1}{2}\varphi_\varepsilon * \varphi_\varepsilon(0)$$

Pf Key fact: (X,Y) centered normal $\Rightarrow E\left(e^{X - \frac{1}{2}Var(X)} e^{Y - \frac{1}{2}Var(Y)} \right)$

This is why normal ordering regularizes.

$$= e^{\frac{1}{2}Var(X+Y) - \frac{1}{2}Var(X) - \frac{1}{2}Var(Y)} = e^{Cov(X,Y)}$$

$$\overset{\text{exp wrt. } W}{E}\left(M_t^\varepsilon M_t^\delta \right) = E^x \otimes E^x\left(\exp\left\{ \int_0^t \varphi_\varepsilon(y - B_{t-r}) \varphi_\delta(y - \tilde{B}_{t-r})\, dr\,dy \right\} \right)$$

$$= E^x \otimes E^x\left(\exp\left\{ \int_0^t \varphi_\varepsilon * \varphi_\delta(B_r - \hat{B}_r)\, dr \right\} \right)$$

$$= E^0\left(\exp\left\{ \int_0^t \varphi_\varepsilon * \varphi_\delta(\sqrt{2} B_r)\, dr \right\} \right)$$

Recall Tanaka formula:

$$\frac{1}{2\varepsilon}\int_0^t \underbrace{\mathbb{1}_{[-\varepsilon,\varepsilon]}(B_s)}_{\text{replace by } \varphi_\varepsilon * \varphi_\delta}\,ds \xrightarrow[\varepsilon \downarrow 0]{P} |B_t| - |B_0| - \int_0^t \operatorname{sgn}(B_s)\,dB_s =: L_t(0)$$

Need

Lemma $\forall p \geq 1 \quad \sup_{0<\varepsilon<1} E^0\left(\exp\left\{p\int_0^t \varphi_\varepsilon * \varphi_\varepsilon(B_r)\,dr\right\}\right) < \infty.$

Pf Interpret as FKF for $\partial_t u = \partial_{xx} u + p\,\varphi_\varepsilon * \varphi_\varepsilon\, u$

$$\| M_t^\varepsilon - M_t^\delta \|_2^2 = \mathbb{E}(M_t^\varepsilon M_t^\varepsilon) - 2\mathbb{E}(M_t^\varepsilon M_t^\delta) + \mathbb{E}(M_t^\delta M_t^\delta)$$

Know: $\mathbb{E}(M_t^\varepsilon M_t^\delta) \xrightarrow[\varepsilon,\delta \downarrow 0]{} E^0\left(e^{L_t(0)\sqrt{2}}\right).$

Conclusion: FKF holds for CH-KPZ:

$$u(t,x) = E^x\left(u_0(B_t)\,M_t\right) \quad \text{where} \quad M_t = :\exp\left\{\int_0^t \delta(y - B_{t-r})\,W(dr\,dy)\right\}: \;:= \lim_{\varepsilon \downarrow 0} M_t^\varepsilon.$$