Reminder: Trying to solve rough PDEs:

SHE: $\partial_t u = \Delta u + \dot{W}$, $u(0,\cdot) = u_0(\cdot)$

d=1: $u(t,x) = g_t * u_0(x) + \int g_{t-s}(x-y)\, W(ds\,dy)$

KPZ $\partial_t h = \Delta h + |\nabla h|^2 + \dot{W}$

Cole-Hopf: $u(t,x) = e^{h(t,x) - \frac{1}{2}t}$ "solves" $\qquad g_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4t}}$

$$\partial_t u = \Delta u + u\dot{W}$$

$\boxed{d=1}$

$$u(t,x) = g_t * u_0(x) + \sum_{n=1}^{\infty} \int_{t_1 < \cdots < t_n \le t} \underbrace{\prod_{k=2}^{n+1} g_{t_k - t_{k-1}}(x_k - x_{k-1})\; g_{t_1} * u_0(x_1)}_{(t_{n+1} := t,\ x_{n+1} := x)} \prod_{i=1}^{n} W(dt_i\,dx_i)$$

defines $f_n(t_1,x_1,\dots,t_n,x_n)$ on $D_n \times \mathbb{R}^n$

extending symmetrically:

$$= g_t * u_0(x) + \sum_{n=1}^{\infty} \frac{1}{n!} \int f_n \, d{:}W^{\otimes n}{:}$$

Note · look like time-ordered exponential

$$:e^{\int_0^t f(s)\,dB_s}: = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} :\left(\int_0^t f(s)\,dB_s\right)^n: \quad I_t^{(n)}(f\otimes\cdots\otimes f)$$

$$= 1 + \sum_{n=1}^{\infty} \int_0^t \left(\int_0^{t_n} \cdots \left(\int_0^{t_2} f(t_1)\cdots f(t_n)\,dB_{t_1}\right) dB_{t_2}\cdots\right) dB_{t_n}$$

$$\dot{X}(t) = A(t)X(t)$$

$$X(t) = X(0) + \sum_{n=1}^{\infty} \int_{t_1<\cdots<t_n\le t} A(t_n)\cdots A(t_1)X_0\, dt_1\cdots dt_n = T\exp\left\{\int_0^s A(s)\,ds\right\} X_0$$

· Is $u(t,x)>0$ if $u_0>0$?

NB: Only then we can "define" $h(t,x) := \log u(t,x) + \frac{t}{2}$.

NB: $:e^{\int_0^t f(s)\,dB_s}: = e^{\int_0^t f(s)\,dB_s - \frac{1}{2}\int_0^t f(s)^2\,ds} > 0$

Idea: Use Feynman-Kac formula.

<u>Thm</u> (Feynman-Kac formula) Let $u_0 : \mathbb{R}^d \to \mathbb{R}$ be continuous and $f : \mathbb{R}_+ \times \mathbb{R}^d \longrightarrow \mathbb{R}$ continuous & C^2 in 2^{nd} variable.

Assume

$$\exists \eta \in (0,2) \quad \forall t \geq 0 : \quad \sup_{x \in \mathbb{R}^d} \frac{\log(|u_0(x)| \vee 1)}{1+|x|^\eta} < \infty \;\wedge\; \sup_{s \leq t} \sup_{x \in \mathbb{R}^d} \frac{f(s,x)_+}{1+|x|^\eta} < \infty.$$

(up to 2^{nd} derivative)

Then

$$u(t,x) := E^x\left(u_0(B_t) \exp\left\{ \int_0^t f(t-r, B_r)\, dr \right\} \right)$$

is finite, continuous, $C^1(\mathbb{R}_+)/C^2(\mathbb{R}^d)$ and solves

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + u(t,x) f(t,x) & t>0, x \in \mathbb{R}^d \\ u(0,x) = u_0(x) \end{cases}$$

Moreover, u is the unique solution in class of $\tilde{u} : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ subject to

$$\exists \eta \in (0,2) \;\forall t \geq 0 : \; \frac{\log |\tilde{u}(t,x)| \vee 1}{1+|x|^\eta} < \infty.$$

Pf: <u>u is well def'd</u>

Integrand bounded by $e^{C(t)(1+|B_t|^\gamma) + \tilde{C}(t)t \sup_{s\le t}(1+|B_s|^\gamma)}$

Now use $P\left(\sup_{s\le t}|B_s|>a\right)\le d\, e^{-\frac{a^2}{2dt}}$

<u>natural martingale</u>

$$E^x\left(u_0(B_t)\exp\left\{\int_0^t f(t-r,B_r)\,dr\right\}\Big|\ \underset{0\le s<t}{\mathcal{F}_s^B}\right) \overset{MP}{=} \exp\left\{\int_0^s f(t-r,B_r)\,dr\right\} \times \underbrace{E^{B_s}\left(u_0(B_{t-s})\exp\left\{\int_0^{t-s} f(t-s-r,B_r)\,dr\right\}\right)}_{u(t-s,B_s)}$$

So $M_s^{(t)} := u(t-s,B_s)\exp\left\{\int_0^{t-s} f(t-s-r,B_r)\,dr\right\}$ is a martingale.

<u>differentiability</u>

$$u(t,x) = \int dy\ g_t(x-y)\,u_0(y)\,E^0\left(\exp\left\{\int_0^t f(t-r,x+B_r)\,dr\right\}\Big|\ B_t=y-x\right)$$

Brownian bridge: $W_r := B_r - \frac{r}{t}B_t$ $\quad Cov(W_r,B_t) = Cov(B_r,B_t) - \frac{r}{t}Var(B_t) = 0$

$$= \int dy\ g_t(x-y)\,u_0(y)\,E\left(\exp\left\{\int_0^t f\left(t-r,\ x+\frac{r}{t}(y-x)+W_r\right)dr\right\}\right)$$

differentiability in t

$$u(t+\delta,x) = E^x\left(u(t,B_\delta)\exp\left\{\int_0^\delta f(t+\delta-r,B_r)\,dr\right\}\right)$$

$$\frac{u(t+\delta,x)-u(t,x)}{\delta} = \frac{1}{\delta}E^x\big(u(t,B_\delta)-u(t,x)\big)$$
$$+\frac{1}{\delta}E\left(u(t,B_\delta)\left[\exp\left\{\int_0^\delta f(t+\delta-r,B_r)\,dr\right\}-1\right]\right)$$
$$\xrightarrow[\delta\downarrow 0]{} \frac{1}{2}\Delta u(t,x) + u(t,x)f(t,x)$$

Converse If u is a solution in subgaussian class:

$$M_s := u(t-s,B_s)\exp\left\{\int_0^s f(t-r,B_r)\,dr\right\}$$

Itô: $$dM_s = \left[-\frac{\partial u}{\partial t}(t-s,B_s) + \frac{1}{2}\Delta u(t-s,B_s) + f(t-s,B_s)u(t-s,B_s)\right]\exp\{\ldots\}\,ds$$
$$+ \exp\{\ldots\}\nabla u(t-s,B_s)\cdot dB_s$$

So M_s is loc. martingale. Subgauss growth: $|M_s| \le e^{C(t)(1+|B_s|)^3 + \tilde{C}(t)t\sup_{s\le t}[1+|B_s|^3]}$

So M is a martingale.

Then $u(t,x) = E^x(M_0) = E^x(M_t) = E^x\Big(\underbrace{u(0, B_t)}_{= u_0(B_t)} \exp\Big\{ \int_0^t f(t-r, B_r)\,dr \Big\} \Big)$