Solving CH-KPZ

$$\partial_t h(t,x) = \Delta h(t,x) + |\nabla h(x,t)|^2 + \dot{W}(t,x)$$

$$u(t,x) = e^{h(t,x) - \frac{1}{2}t} \Rightarrow \boxed{\partial_t u = \Delta u + u\dot{W}}$$

Cole-Hopf form of KPZ

Need to define $\int Y(x)\, W(dx)$ for Y random

Def We say $\{Y(x) : x \in \mathcal{X}\}$ is W-integrable if
$\forall n \geq 0 \; \exists f_n \in L^{2,\mathrm{sym}}(\mathcal{X}^{n+1}, \mathcal{G}^{\otimes n+1}, \mu^{\otimes n+1})$ s.t.

$$Y(x) = f_0(x) + \sum_{n=1}^{\infty} \int f_n(x, \cdot)\, d{:}W^{\otimes n}{:} \quad \text{for } \mu\text{-a.e. } x.$$

where $\sum_{n \geq 0} \|f_n\|^2_{L^2(\mu^{\otimes n+1})} < \infty$. Then we say

$$\int Y(x)\, W(dx) := \sum_{n \geq 0} \int f_n \, dW^{\otimes(n+1)}$$

Solution by iteration:
(d=1)

$$\boxed{\begin{aligned} \partial_t h(t,x) &= \Delta h(t,x) + |\nabla h(x,t)|^2 + \dot{W}(t,x) \\ u(t,x) &= e^{h(t,x) - \frac{1}{2}t} \Rightarrow \boxed{\partial_t u = \Delta u + u\dot{W}} \end{aligned}}$$

Cole-Hopf form of KPZ

$$u(t,x) = \int_{[0,t]\times\mathbb{R}} g_{t-s}(x-y)\,u_0(y)\,ds\,dy + \int_{[0,t]\times\mathbb{R}} g_{t-s}(x-y)\,u(s,y)\,W(ds\,dy)$$

$$= \int_{[0,t]} g_{t-s}(x-y)\,u_0(y) + \dots$$

$$\dots + \int_{\substack{0<t_1<\dots<t_n\le t \\ x_1\dots x_n\in\mathbb{R}}} g_{t-t_n}(x-x_n)\,g_{t_n-t_{n-1}}(x_n-x_{n-1})\cdots g_{t_1} * u_0(x_1)\,W(dt_1dx_1)\dots W(dt_ndx_n)$$

$$+ \dots$$

<u>Idea</u>: Take $n\to\infty$ and write $u(t,x)$ as infinite series.

Lemma $\exists C>0\ \forall t\geq 0\ \forall x\in\mathbb{R}\ \forall n\geq 1$:

$$\int_{\substack{0<t_1<\dots<t_n\leq t\\ x_1,\dots,x_n\in\mathbb{R}}} \prod_{k=2}^{n+1} g_{t_k-t_{k-1}}(x_k-x_{k-1})^2 \; g_{t_1}*u_0(x_1)^2 \leq \frac{C^n t^{\frac{n-1}{2}}}{\Gamma(\frac{n}{2}+1)} \|u_0\|^2$$

Here: $x_{n+1}:=x$, $t_{n+1}:=t$.

Pf: Bound $g_s(x)^2\leq \frac{1}{\sqrt{4\pi s}}\, g_s(x)$ to dominate LHS by:

$$\int_{0<t_1<\dots<t_n\leq t} \prod_{k=1}^{n+1} \frac{1}{\sqrt{4\pi(t_k-t_{k-1})}} \|u_0\|^2$$

$$\underset{u_i=\sqrt{t_i-t_{i-1}}}{\leq} \frac{1}{\sqrt{4\pi t}} (4\pi)^{-\frac{n+1}{2}} \|u\|^2\, 2^{-n} \int_{u_1^2+\dots+u_n^2\leq t} du_1\dots du_n$$

$$\leq \frac{1}{\sqrt{4\pi t}} (4\pi)^{-\frac{n+1}{2}} \|u\|^2\, 2^{-n}\, t^{n/2}\, \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$$

used:

$$\int g_{t_1}*u_0(x_1)^2\,dx_1 \leq \int g_{t_1}(x_1)^2\,dx_1\, \|u_0\|^2$$

$$\int \prod_{k=1} g_{t_k-t_{k-1}}(x_k-x_{k-1}) \prod_{k=1}^{n} dx_k = g_t(x) \leq \frac{1}{\sqrt{4\pi t}}$$

So we showed:

$$u(t,x) := g_t * u_0(x) + \sum_{n=1}^{\infty} \int_{\substack{0<t_1<\cdots<t_n\le t \\ x_1,\dots,x_n\in\mathbb{R}}} \left(\prod_{k=2}^{n+1} g_{t_k-t_{k-1}}(x_k-x_{k-1})\right) g_{t_1} * u_0(x_1)\, W(dt_1 dx_1)\cdots W(dt_n dx_n)$$

is well defined with sum convergent in L^2.

Lemma (Mixed Fubini) Let $(\mathcal{X},\mathcal{G},\mu)$, $(\mathcal{Y},\mathcal{H},\nu)$ be finite meas. spaces, let W = white noise on $(\mathcal{X},\mathcal{G},\mu)$. Then $\forall n\ge 1$ $\forall f\in L^2(\mathcal{X}^n,\mathcal{G}^{\otimes n},\mu^{\otimes n})$ there exists a measurable version of

$$y \longmapsto \int f(\cdot,y)\, dW^{\otimes n}$$

and $\forall h\in L^2(\mathcal{Y},\mathcal{H},\nu)$:

$$\int h(y)\left(\int f(\cdot,y)\, dW^{\otimes n}\right)\nu(dy) = \int\left(\int h(y) f(\cdot,y)\,\nu(y)\right) dW^{\otimes n}$$

Corollary (Minkowski + isometry)

$$E\left(\left[\int h(y)\left(\sum_{k=m}^{n} \int f_k(\cdot, y)\, dW^{\otimes k}\right) \nu(dy)\right]^2\right)$$
$$\leq \|h\|^2_{L^2(\nu)} \sum_{k=m}^{n} \|f_k\|^2_{L^2(\mu^{\otimes n} \otimes \nu)}$$

This shows that integrals with test functions can be passed inside the series and inside PW integrals. This yields:

$$\int u(t,x)\, h(t,x)\, dt\, dx = \int g_t * u_0(x)\, h(t,x)\, dt\, dx$$
$$+ \int h(t,x)\, g_{t-s}(x-y)\, u(s,y)\, W(ds\, dy)\, dt\, dx$$

So u solves integral form eq. in weak sense. Now replace h by $(-\partial_t - \partial_{xx})h$ & integrate by parts. $\boxtimes$