Quadratic variation process

Thm Let $\{M_t : t \geq 0\}$ be continuous local martingale w.r.t. filtration $\{\mathcal{F}_t\}_{t\geq 0}$ s.t $\mathcal{F}_0$ contains all P-null sets. Then $\exists$ $\{\langle M\rangle_t : t\geq 0\}$ an adapted continuous, non-decreasing process with $\langle M\rangle_0 = 0$ s.t.

$$\{ M_t^2 - \langle M\rangle_t : t \geq 0\} \text{ is local martingale}$$

Ex $M_t = B_t$, $\langle M\rangle_t = t$

$M_t = \int_0^t Y_s \, dB_s$, $\langle M\rangle_t = \int_0^t Y_s^2 \, ds$

Uniqueness last time, existence by $\langle M\rangle_t$ "$:=$" $\lim_{n\to\infty} V_t^{(2)}(M,\Pi_n)$

Lemma Suppose $\exists K \geq 0$: $\sup_{t\geq 0} |M_t| \leq K$. Then $\forall t \geq 0$ $\forall \Pi, \widehat{\Pi}$ = partitions of $[0,t]$ with $\|\widehat{\Pi}\| \leq \|\Pi\|$,

$$E\left(\left|V_t^{(2)}(M,\Pi) - V_t^{(2)}(M,\widehat{\Pi})\right|^2 \right) \leq 112\, K^2 \left[E\left(\operatorname{osc}_M([0,t], \|\Pi\|)^4 \right) \right]^{1/2}$$

Pf: Assume first $\tilde{\Pi}$ is refinement of Π. Label these as

$$\Pi = \{ t_{i,0} : i = 0 \cdots n+1 \}, \quad \tilde{\Pi} = \{ t_{i,j} : i = 1, \dots, n,\ j = 0 \cdots m_i \}.$$

s.t. $0 = t_{1,0} < t_{1,1} \cdots < t_{1,m_1} = t_{2,0} < \cdots < t_{n-1,m_{n-1}} = t_{n,0} < \cdots < t_{n,m_n} = t_{n+1,0} = t$

Then $$V_t^{(2)}(M, \tilde{\Pi}) = \sum_{i=1}^{n} \sum_{j=1}^{m_i} Z_{i,j}^2$$ where $Z_{i,j} := M_{t_{i,j}} - M_{t_{i,j-1}}$

and $$V_t^{(2)}(M, \Pi) = \sum_{i=1}^{n} \left(\sum_{j=1}^{m_i} Z_{i,j} \right)^2$$

So $$V_t^{(2)}(M, \Pi) - V_t^{(2)}(M, \tilde{\Pi}) = 2 \sum_{i=1}^{n} \sum_{1 \le j < k \le m_i} Z_{i,j} Z_{i,k}$$

and $$E\left(\left[\ \text{— '' —}\ \right]^2\right) = 4 \sum_{i=1}^{n} \sum_{i'=1}^{n} \sum_{1 \le j < k \le m_i} \sum_{1 \le j' < k' \le m_{i'}} \underbrace{E\left(Z_{i,j} Z_{i,k} Z_{i',j'} Z_{i',k'}\right)}_{= 0 \text{ unless } i = i',\ k = k'}$$

by fact that Z's are martingale increments

So $$E([\ldots]^2) = 4\sum_{i=1}^{n}\sum_{k=2}^{m_i}\sum_{j,j'=1}^{k-1} E(Z_{i,j} Z_{i,j'} Z_{i,k}^2)$$

$$= 4\sum_{i=1}^{n}\sum_{k=2}^{m_i} E\Big(\underbrace{(M_{t_{i,k-1}} - M_{t_{i,0}})^2}_{\le \text{osc}} (M_{t_{i,k}} - M_{t_{i,k-1}})^2\Big)$$

$$\le 4E\Big(\text{osc}_M([0,t], \|\pi\|)^2\, V_t^{(2)}(M, \tilde{\pi})\Big)$$

Now separate terms using Cauchy-Schwarz and apply

<u>Lemma</u> For M martingale with $\sup_{t\ge 0}|M_t| \le K$ we have

$$\boxed{E\left(V_t^{(2)}(M,\pi)^2\right) \le 48K^4} \le (7K^2)^2$$

Lemma Under conditions of prev. lemma, there exists $\{\langle M\rangle_t : t \geq 0\}$ adapted, continuous, non-decreasing with $\langle M\rangle_0 = 0$ s.t. $\forall t \geq 0$ $\forall \Pi_n$ = partitions of $[0,t]$:

$$\|\Pi_n\| \to 0 \implies \bar{V}_t^{(2)}(M, \Pi_n) \xrightarrow{L^2} \langle M\rangle_t$$

Moreover, $E(\langle M\rangle_t) \leq 4K^2$.

Pf. Some general considerations: Let $0 \leq u < s < t$. Then

$$E((M_t - M_u)^2 \mid \mathcal{F}_s) \underset{\text{martingale}}{=} E((M_t - M_s)^2 \mid \mathcal{F}_s) + (M_s - M_u)^2$$

So if Π is partition of $[0,t]$ and $s \in [0,t)$, then

$$E\left(\bar{V}_t^{(2)}(M, \Pi) \mid \mathcal{F}_s\right) = V_s^{(2)}(M, \Pi') + E((M_t - M_s)^2 \mid \mathcal{F}_s)$$

where Π' is restriction of Π to $[0,s]$. So denoting

$$A_s^{\Pi} := \bar{V}_s^{(2)}(M, \Pi') \quad \text{for } \Pi' = \text{restriction of } \Pi \text{ to } [0,s]$$

for any two partitions $\Pi, \tilde{\Pi}$ of $\mathbb{R}_+$ we get

$$A_s^{\Pi} - A_s^{\tilde{\Pi}} = E\left(A_t^{\Pi} - A_t^{\tilde{\Pi}} \mid \mathcal{F}_s \right)$$ meaning that

$\{ A_s^{\Pi} - A_s^{\tilde{\Pi}} : s \geq 0 \}$ is a martingale

Moreover, by continuity of M, $t \mapsto A_t^{\Pi}$ is continuous!

Doob's L^2-inequality gives

$$E\left(\sup_{s \leq t} |A_s^{\Pi} - A_s^{\tilde{\Pi}}|^2 \right) \leq 4 E\left(|A_t^{\Pi} - A_t^{\tilde{\Pi}}|^2 \right)$$

$$\overset{\text{Lemma}}{\leq} 4 \cdot 112 \cdot K^2 \left[E\left(\mathrm{osc}_M\left([0,t], \max(\|\Pi\|, \|\tilde{\Pi}\|)\right)^4 \right) \right]^{1/2}$$

Now choose $\{\Pi_n\}$ s.t. $\|\Pi_{n+1}\| \leq \|\Pi_n\|$ and

$$E\left(\mathrm{osc}_M\left([0,n], \|\Pi_n\|\right)^4 \right) \leq 64^{-n}$$

Chebyshev:

$$P\Big(\sup_{s\le n} |A_s^{\Pi_n} - A_s^{\Pi_{n+1}}| > 2^{-n} \Big) \le 4 \cdot 112 \cdot K^2 \, 2^{-n}$$

Let $\Omega^* = \Omega \setminus \{ \sup_{s \le n} |A_s^{\Pi_n} - A_s^{\Pi_{n+1}}| > 2^{-n} \text{ i.o.} \}$. BC $\Rightarrow P(\Omega^*) = 1$

Then set

$$A_t := \begin{cases} \lim_{n\to\infty} A_t^{\Pi_n} & \text{on } \Omega^* \\ 0 & \text{on } \Omega \setminus \Omega^* \end{cases}$$

By uniformity of limit, A is continuous, $A_0 = 0$.

Lemma above then shows $\forall t \ge 0$: $V_t^{(2)}(M, \Pi) \xrightarrow[\|\Pi\| \to 0]{L^2} A_t$

Taking Π_n containing s, we get

$$V_t^{(2)}(M, \Pi_n) - V_s^{(2)}(M, \Pi_n) \ge 0$$

we get $A_t \ge A_s$ a.s. Set $\Omega_0 := \bigcap_{\substack{s,t \in \mathbb{Q} \\ s<t}} \{A_s \le A_t\}$, then $P(\Omega_0) = 1$

Then $\langle M \rangle_t := \begin{cases} A_t & \text{on } \Omega_0 \\ 0 & \text{on } \Omega \setminus \Omega_0 \end{cases}$ is non-decreasing. $\boxtimes$

Pf of Thm: Assume first $\sup_{t \ge 0} |M_t| \le K$.

Then $\langle M \rangle$ above exists and we have

$$E(M_t^2 | \mathcal{F}_s) - M_s^2 = E((M_t - M_s)^2 | \mathcal{F}_s)$$

$$\underset{\substack{s \text{ belongs} \\ \text{to } \Pi}}{=} E\left(V_t^{(2)}(M, \Pi) - V_s^{(2)}(M, \Pi) \,\middle|\, \mathcal{F}_s \right)$$

Taking $\|\Pi_n\| \to 0$ gives

$$E(M_t^2 | \mathcal{F}_s) - M_s^2 = E\left(\langle M \rangle_t - \langle M \rangle_s \,\middle|\, \mathcal{F}_s \right) = E(\langle M \rangle_t | \mathcal{F}_s) - \langle M \rangle_s .$$

So $\{ M_t^2 - \langle M \rangle_t : t \ge 0 \}$ is a martingale.

For local martingale, truncate by stopping time $\tau_K := \inf\{ t \ge 0 : |M_t| \ge K \}$.

Then $\{ M_{\tau_K \wedge t} : t \ge 0 \}$ is proper martingale. So $\exists \langle M \rangle^{(K)}$ s.t.

$\{ M^2_{\tau_K \wedge t} - \langle M \rangle^{(K)}_t : t \ge 0 \}$ is martingale. For $K \le L$,

$\{ \langle M \rangle^{(K)}_{\tau_K \wedge t} - \langle M \rangle^{(L)}_{\tau_K \wedge t} : t \ge 0 \}$ is loc. martingale with BV. Uniqueness:

$\forall t \le \tau_K : \langle M \rangle^{(K)}_t = \langle M \rangle^{(L)}_t$. Define $\langle M \rangle_t := \begin{cases} \lim_{K \to \infty} \langle M \rangle^{(K)}_t & \text{on } \{\tau_K \to \infty\} \\ 0 & \text{else.} \end{cases}$ $\boxtimes$