Stochastic heat eq.

$\partial_t u = \partial_{xx} u + \dot{W}$, $g_t(x) = \frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/4t}$

$$u^\varepsilon(t,x) := g_t * u_0(x) + \int_0^t g_{t-s}(x-y)\, \varphi_\varepsilon(s,y)\, W(ds\,dy)$$

as $\varepsilon \downarrow 0$ this converges ($\varphi \geq 0$, $\varphi \in C_c$, $\int \varphi = 1$) to

$$u(t,x) := g_t * u_0(x) + \int_0^t g_{t-s}(x-y)\, W(ds\,dy) \quad (\text{ptws a.s})$$

we checked that u is Hölder cont. ($\frac{1}{4}-$ in time, $\frac{1}{2}-$ in space)

WTS: u is weak solution to SHE.

Lemma Let $h \in L^2(\mathbb{R}_+ \times \mathbb{R}^d)$ with support in $[-R,R] \times [-R,R]^d$

Then $X_h := \int h(x,t)\, g_{t-s}(x-y)\, W(ds\,dy)$

is well defined and obeys $\|X_h\|_{L^2(P)} \leq C_R \|h\|_{L^2([-R,R]^{d+1})}$

Pf $\quad E(X_n^2) = \int h(x,t)\, h(x',t')\, C(x,t;x',t')\, dx\, dx'\, dt\, dt'$

$$C(x,t;x',t') = A_d \int_{|t-t'|}^{t+t'} \frac{1}{r^{d/2}}\, e^{-\frac{|x-x'|^2}{4r}}\, dr \qquad A_d = \frac{1}{2(4\pi)^{d/2}}$$

$$E(X_n^2) \overset{(h\geq 0)}{\leq} A_d \int \frac{1}{r^{d/2}}\, dr \int_{[-R,R]^{2(d+1)}} h(x,t)\, h(x',t') \left(\mathbb{1}_{\|x-x'\|_\infty \leq r^{\frac{1}{2}-\frac{1}{2d}}}\, \mathbb{1}_{|t-t'|<r} + \mathbb{1}_{|x-x'|> r^{\frac{1}{2}-\frac{1}{2d}}}\, e^{-\frac{|x-x'|^2}{4r}} \right) dx\, dx'\, dt\, dt'$$

$$\leq A_d \int_0^{2R} \frac{1}{r^{d/2}}\, dr \left[\underbrace{\left(2\, r^{\frac{1}{2}-\frac{1}{2d}}\right)^d}_{r^{\frac{d}{2}-\frac{1}{2}}} 2R\, \|h\|_2^2 + (2R)^{d+1} \|h\|^2 e^{-r^{-\frac{1}{d}}/4} \right]$$

$$\leq C_R \|h\|^2 \qquad \boxtimes$$

Notice $\quad X_{h*\varphi_\varepsilon} = \int \left(\int h(x,t)\, g_{t-s}(x-y)\, \varphi_\varepsilon(s-u, y-z)\, dx\, dy\, dt\, ds \right) W(dz\, du)$

$$= \int h(x,t)\, u^\varepsilon(t,x)\, dt\, dx$$

Proof of $u^\varepsilon \to u$ weakly (in L^2) & u being weak solution

$h \in C_c(\mathbb{R}_+ \times \mathbb{R})$, then $h * \varphi_\varepsilon \to h$ in $L^2(\mathbb{R}_+ \times \mathbb{R})$

So $\int h u^\varepsilon = X_{h * \varphi_\varepsilon} \xrightarrow{L^2} X_h = \int h u$

If $h \in C_c^2(\mathbb{R}_+ \times \mathbb{R})$

$$\int (-\partial_t - \partial_{xx}) h \, u^\varepsilon \longrightarrow \int (-\partial_t - \partial_{xx}) h \, u$$

$$\int h \dot{W}_\varepsilon \xrightarrow[L^2]{\text{Lemma}} \int h \, dW$$

$C(t,x,t,x') \sim \frac{1}{\|x-x'\|^{d-2}}$ $d \geq 3$

$\sim \log\|x-x'\|$ $d=2$

Gaussian Free Field at short distances

Lemma: $\forall h \in L^2(\mathbb{R}_+ \times \mathbb{R}^d)$, supp$(h)$ compact:

$$\int h(t,x) u^\varepsilon(t,x) \, dt \, dx \xrightarrow[\varepsilon \downarrow 0]{L^2} X_h$$

converge in distribution theory.

where $\{X_h : h \in L^2(\mathbb{R}_+ \times \mathbb{R}^d)\}$ is Gaussian process with covariance kernel $C(x,t,x',t')$.

KPZ

$$\partial_t h = \Delta h + |\nabla h|^2 + \dot W$$

Cole-Hopf version:

(Brownian sheet)

write as SDE: $$dh = (\Delta h + |\nabla h|^2)\,dt + dB_t(x)$$

$$u(t,x) = e^{h(t,x) - \frac{1}{2}t}$$

Itô: $$du = e^{h-\frac{1}{2}t}\left(dh - \tfrac{1}{2}dt\right) + \tfrac{1}{2} e^{h-\frac{1}{2}t}\,dt$$

$$= \underbrace{e^{h-\frac{1}{2}t}(\Delta h + |\nabla h|^2)}_{\Delta e^{h-\frac{1}{2}t}}\,dt + e^{h-\frac{1}{2}t}\,dB_t(x)$$

So u "solves" $$\boxed{\partial_t u = \Delta u + u W}$$

SHE w/ multiplicative noise

Parabolic Anderson model with space-time white noise

Cole-Hopf KPZ eq.