Solving Stochastic heat equation

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + \dot{W} \\ u(0,x) = u_0(x) \end{cases}$$

$\dot{W}$ ← space-time white noise

$\int_A \dot{W}(t,x)\,dt\,dx = W(A)$ ← white noise on $(\mathbb{R}_+ \times \mathbb{R}^d, \text{Borel}, \text{Leb})$

key issue: interpretation of "solve"

- classical solution - meaningless
- weak solution
- mollify the noise term

$\varphi : \mathbb{R}_+ \times \mathbb{R}^d \to [0,\infty)$, Borel, bdd, compact support

$$\varphi_\varepsilon(t,x) = \frac{1}{\varepsilon^{d+1}} \varphi\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$$

$$\dot{W}_\varepsilon(t,x) := \int \varphi_\varepsilon(t-s, x-y)\, W(ds\,dy) = \int \varphi_\varepsilon(t-\cdot, x-\cdot)\, dW$$

Lemma 1) $\varphi \in L^2 \Rightarrow \{\dot{W}_\varepsilon(t,x) : t \geq 0, x \in \mathbb{R}^d\}$ well def.

2) $\varphi \in C_c(\mathbb{R}_+ \times \mathbb{R}^d)$ is γ-Hölder $\Rightarrow \{\dot{W}(t,x) : t \geq 0, x \in \mathbb{R}^d\}$ is γ'-Hölder for $\gamma' < \gamma$

3) $\forall h \in L^2(\mathbb{R}_+ \times \mathbb{R}^d)$: $\int h(t,x)\, \dot{W}_\varepsilon(t,x)\, dt\,dx \xrightarrow[\varepsilon \downarrow 0]{} \int h\, dW$

Pf: 1) def of Paley-Wiener integral

2) $$E\left(|\dot W_\varepsilon(t,x) - \dot W_\varepsilon(t',x')|^2\right) = \int |\varphi_\varepsilon(t-s, x-y) - \varphi_\varepsilon(t'-s, x'-y)|^2 \, ds\, dy$$
$$\leq C(\varepsilon)\left(|t-t'|^{2\gamma} + \|x-x'\|^{2\gamma}\right)$$

Gaussianness:
$$E\left(|\dot W_\varepsilon(t,x) - \dot W_\varepsilon(t',x')|^{\overset{2n}{\uparrow \alpha}}\right) \leq C(n,\varepsilon)\left(|t-t'|^{\overset{2n\gamma}{\uparrow d+\beta}} + \|x-x'\|^{2n\gamma}\right)$$

Kolmogorov-Čentsov: $\dot W_\varepsilon$ admits γ'-Hölder continuous version $\forall \gamma' < \beta/\alpha = \frac{2n\gamma - d}{2n} = \gamma - \frac{d}{2n}$

3) $$E\left(\left|\int h \dot W_\varepsilon \, dt\, dx - \int h\, dW\right|^2\right) = E\left(\left|\int \varphi_\varepsilon * h\, dW - \int h\, dW\right|^2\right)$$
$$= \int |\varphi_\varepsilon * h(t,x) - h(t,x)|^2 \, dt\, dx \xrightarrow[\varepsilon \downarrow 0]{\int \varphi_\varepsilon = 1} 0$$

add to statement?

$\boxtimes$

Note $\dot W_\varepsilon(t,x) = \varphi_\varepsilon * \dot W(t,x)$

Lemma Let $f: \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ continuous, at most exp growth.

$u_0: \mathbb{R}^d \to \mathbb{R}$ continuous, — " —

Then
$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + f(t,x) & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d \\ u(0,x) = u_0(x) & x \in \mathbb{R}^d \end{cases}$$

is solved by

$$u(t,x) := g_t * u_0(x) + \int_0^t g_{t-s} * f(s,\cdot)(x)\, ds$$

for $g_t(x) := \frac{1}{\sqrt{4\pi t}} e^{-|x|^2/4t}$. Moreover, the solution is unique among functions with at most exponential growth.

Pf $(\partial_t - \Delta) g_t(x) = 0$, $g_t(x)\,dx \xrightarrow[t \downarrow 0]{} \delta_0(dx)$

So our mollified PDE is solved by

$$u^{\varepsilon}(t,x) := g_t * u_0(x) + \int_0^t ds \int dy \; g_{t-s}(x-y)\, \dot{W}_{\varepsilon}(s,y)$$

Want: $\varepsilon \downarrow 0$.

<u>Q:</u> Is $\int_0^t ds \int dy \; g_{t-s}(x-y)^2 < \infty$ (Needed for limit to be defined).

$$\text{integral} = \int_0^t ds \int dy \frac{1}{(4\pi s)^d} e^{-\frac{|y|^2}{2s}} = \frac{1}{(4\pi)^d}\underbrace{\left(\int_0^t \frac{1}{s^{d/2}} ds\right)}_{<\infty \text{ iff } d=1 \,!}\left(\int dy\, e^{-\frac{|y|^2}{2}}\right)$$

<u>Thm:</u> Assume $d=1$. Then $\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}$: $u^{\varepsilon}(t,x) \xrightarrow[\varepsilon \downarrow 0]{P} u(t,x)$

where $u(t,x) := g_t * u_0(x) + \int g_{t-s}(x-y)\, W(ds\, dy)$

which is well defined and admits a cont. version.

Moreover, $\{u(t,x) : t \geq 0, x \in \mathbb{R}\}$ is a weak solution to SHE.

Pf Convergence follows from Lemma. Key pt to prove: continuity of u.

WLOG: $u_0 = 0$:

$$E\big(u(t,x)\,u(t',x')\big) = \int_0^{t\wedge t'} ds \int dy\; g_{t-s}(x-y)\, g_{t'-s}(x'-y)$$

$$= \frac{1}{4\pi} \int_0^{t\wedge t'} ds \int dy\; \frac{1}{\sqrt{(t-s)(t'-s)}}\, e^{-\frac{(x-y)^2}{4(t-s)} - \frac{y^2}{4(t'-s)}}$$

$$= \frac{1}{4\pi} \int_0^{t\wedge t'} \frac{1}{\sqrt{t+t'-2s}}\, e^{-\frac{1}{4}\frac{x^2}{t+t'-2s}}\, ds$$

$$= \frac{1}{8\pi} \int_{|t-t'|}^{t+t'} \frac{1}{\sqrt{r}}\, e^{-\frac{1}{4}\frac{x^2}{r}}\, dr$$

Hence:

$$E\big(|u(t,x)-u(t',x)|^2\big) = \int_0^{t} + \int_0^{t'} - 2\int_{|t-t'|}^{t+t'} = \frac{2}{8\pi}\int_0^{|t-t'|} \frac{1}{\sqrt{r}}\,dr + \frac{2}{8\pi}\int_{t\wedge t'}^{t\vee t'} \frac{1}{\sqrt{r}}\,dr \le \frac{1}{2\pi}|t-t'|^{1/2}$$

$$E\big(|u(t,x)-u(t,x')|^2\big) \le \frac{2}{\sqrt{4\pi}} \int_0^{t} \frac{1}{\sqrt{r}}\left(1-e^{-|x-x'|^2/4r}\right) dr \le C|x-x'|$$