Expansion in iterated Itô integrals

Thm Let $B = \text{SBM}$ on $(\Omega, \mathcal{F}, P)$, $\{\mathcal{F}_t\}_{t\geq 0}$ Brownian filtration, $\mathcal{F}_0 \supseteq \{P\text{-null sets}\}$.

Then $\forall n \geq 1 \ \forall f \in L^{2,loc}(D_n) \ \exists \{I_t^{(n)}(f) : t \geq 0\} \in \mathcal{M}_2^{cont}$

s.t.

1) $\forall f \in L^{2,loc}(D_1)$: $\quad I_t^{(1)}(f) = \int_0^t f(s)\, dB_s$

2) $\forall n \geq 2 \ \forall f \in L^{2,loc}(D_n) \ \exists Y \in \mathcal{V}_B$:

$$\forall t \geq 0: \ Y_t = I_t^{(n-1)}(f_t) \text{ a.s.} \ \wedge \ I_t^{(n)}(f) = \int_0^t Y_s\, dB_s \text{ a.s.}$$

Moreover: $\forall n \geq 1 \ \forall f \in L^2(D_n) \ \forall t \geq 0$: $\quad E\left(I_t^{(n)}(f)^2\right) = \int_{D_n \cap [0,t]^n} f(t_1 \ldots t_n)^2\, dt_1 \ldots dt_n$

Pf: f simple: $\quad 0 \leq s_1 < s_2 < \ldots < s_m$

$$f(t_1 \ldots t_n) = \sum_{1 \leq j_1 < \ldots < j_n \leq m} a_{j_1 \ldots j_n} \prod_{k=1}^{n} \mathbb{1}_{(s_{j_k-1}, s_{j_k}]}(t_k)$$

$$I_t^{(n)}(f) := \sum_{1 \leq j_1 < \ldots < j_n \leq m} a_{j_1 \ldots j_n} \prod_{k=1}^{n} \left(B_{t \wedge s_{j_k}} - B_{t \wedge s_{j_k-1}}\right)$$

Then $\{I_t^{(n)}(f) : t \geq 0\} \in \mathcal{M}_2^{cont}$, $\quad E\left(I_t^{(n)}(f)^2\right) = \int_{D_n \cap [0,t]^n} f(t_1 \ldots t_n)^2\, dt_1 \ldots dt_n$

Fact: simple functions are dense in $L^{2,loc}(D_n)$.

Take $f \in L^{2,loc}(D_n)$, find $\{f^{(k)}\}_{k\geq 0}$ simple s.t. $\forall t \geq 0: \|f - f^{(k)}\|_{L^2(D_n\cap[0,t])} \leq 4^{-k}\lceil t \rceil$

Then $$\int_0^t E\left(\left|I_s^{(n-1)}(f^{(k)}) - I_s^{(n-1)}(f^{(k+1)})\right|^2\right) ds \overset{\text{Itô iso}}{=} \int_0^t \|f^{(k)} - f^{(k+1)}\|^2_{L^2(D_{n-1}\cap[0,s]^{n-1})} ds$$

$$= \|f^{(k+1)} - f^{(k)}\|^2_{L^2(D_n \cap [0,t]^n)} \leq \lceil t \rceil^2 4 \cdot 16^{-k} = C\, 16^{-k}$$

So by Chebyshev:

$$P\left(\lambda\left(s \leq t : \left|I_s^{(n-1)}(f^{(k)}) - I_s^{(n-1)}(f^{(k+1)})\right| > 2^{-k}\right) > 2^{-k}\right) \leq C\, 2^{-k}$$

Borel Cantelli: $\exists\, \Omega^* \in \mathcal{F}_\infty : P(\Omega^*) = 1$ and

$$\forall \omega \in \Omega^* : I_s^{(n-1)}(f^{(k)})(\omega) \xrightarrow[k\to\infty]{} Y_s(\omega) := \begin{cases} \limsup_{k\to\infty} I_s^{(n-1)}(f^{(k)}) & \text{if finite} \\ 0 & \text{else} \end{cases}$$

Set $Y_s(\omega) = 0$ on $\Omega \setminus \Omega^*$: defines $Y \in \mathcal{V}_B$.

So $$\int_0^t Y_s\, dB_s \overset{\text{Itô iso}}{=} L^2\text{-}\lim_{k\to\infty} \int_0^t I_s^{(n-1)}(f_s^{(k)})\, ds = L^2\text{-}\lim_{k\to\infty} I_t^{(n)}(f^{(k)}) =: I_t^{(n)}(f) \quad \text{a.s.}$$

Thm: For $t \geq 0$ define $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$ as follows: $\mathcal{H}_0 = \{\lambda : \lambda \in \mathbb{R}\}$

and $\forall n \geq 1$: $\mathcal{H}_n := \left\{ I_t^{(n)}(f) : f \in L^2(D_n \cap [0,t]^n) \right\}$.

Then $\forall m,n \in \mathbb{N}$: $m \neq n \Rightarrow \mathcal{H}_m \perp \mathcal{H}_n$.

and

$$L^2(\Omega, \tilde{\mathcal{F}}_t^B, P) = \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$$

Pf: Orthogonality: By polarization

$$E\left(I_t^{(m)}(f)\, I_t^{(n)}(g) \right) = \delta_{m,n} \int_{D_n \cap [0,t]^n} f(t_1 \dots t_n)\, g(t_1 \dots t_n)\, dt_1 \dots dt_n.$$

Key issue: $Y \in L^2(\Omega, \tilde{\mathcal{F}}_t^B, P)$: $Y \perp \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n \Rightarrow Y = 0$.

Suffices to show $\{ B_s^k : s \leq t, k \geq 0 \} \subseteq \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$ (by previous proof)

$$d(B_s^k) = \underbrace{k B_s^{k-1}\, dB_s}_{\in \bigoplus_{n=0}^{k-1} \mathcal{H}_n} + \underbrace{\tfrac{1}{2} k(k-1) B_s^{k-2}\, ds}_{\in \bigoplus_{n=0}^{k-1} \mathcal{H}_n} \quad \text{(by induction)}.$$

Corollary (Itô chaos decomposition)

$\forall t \geq 0 \ \forall Y \in L^2(\Omega, \widetilde{\mathcal{F}}_t^B, P) \ \forall n \geq 1 \ \exists f_n \in L^2(D_n \cap [0,t]^n)$:

$$Y = EY + \sum_{n=1}^{\infty} I_t^{(n)}(f_n).$$

Corollary: $\forall t \geq 0 \ \forall Y \in L^2(\Omega, \widetilde{\mathcal{F}}_t^B, P) \ \exists Z \in \mathcal{V}_B$:

$$Y = EY + \int_0^t Z_s \, dB_s.$$

Corollary: $\forall t \geq 0 \ \forall n \geq 1$: $\quad I_t^{(n)}(1_{D_n}) = \frac{1}{n!} t^{n/2} h_n\left(\frac{B_t}{\sqrt{t}}\right)$

n-th Hermite polynomial

$h_n(x) = x^n + \dots$

Pf idea: $I_t^{(n)}(1_{D_n}) = \frac{1}{n!} :B_t^n:$

Ex Solve SDE: $dX_t = X_t \, dB_t$.

Itô chaos dec: (strong solution) $X_t = \underbrace{EX_0}_{EX_t = EX_0 \text{ because } X \in \mathcal{M}_2^{cont}} + \sum_{n \geq 1} I_t^{(n)}(f_n)$?

Plugin SDE:

$$X_t = EX_0 + \int_0^t EX_0 \, dB_s + \sum_{n \geq 1} \underbrace{\int_0^t I_s^{(n)}(f_n) \, dB_s}_{I_t^{(n+1)}(f_n)}.$$

Uniqueness: $f_1 = EX_0 \wedge \forall n \geq 1: f_{n+1} = f_n$.

$$\Rightarrow X_t = EX_0 + \sum_{n \geq 1}^{\infty} I_t^{(n)}(1_{D_n}) \cdot EX_0 = EX_0 \Big[1 + \sum_{n \geq 1} \underbrace{t^{n/2} h_n(B_t/\sqrt{t})}_{t^{n/2} : \frac{B_t^n}{t^{n/2}} :} \frac{1}{n!} \Big]$$

$$X_t = EX_0 \, e^{B_t - \frac{t}{2}}.$$

which can be checked directly.