Iterated Itô integrals

last time: Wick-ordered product: $\;:Y_1 \cdots Y_n: \; := \mathrm{proj}_{\mathcal{H}_n}(Y_1 \cdots Y_n)$

$(\mathcal{H}_1^{\,n} \longrightarrow \mathcal{H}_n)$

$$f = \sum_{i_1 \ldots i_n = 1}^{m} a_{i_1 \ldots i_n} \mathbb{1}_{A_{i_1} \times \cdots \times A_{i_n}} \qquad A_1 \ldots A_m \text{ disjoint}$$

$$\int f \, d{:}W^{\otimes n}{:} \; := \sum_{i_1 \ldots i_n = 1}^{m} a_{i_1 \ldots i_n} :W(A_{i_1}) \cdots W(A_{i_n}):$$

$$L^2_{sym}(\mathcal{X}^n, \mathcal{G}^{\otimes n}, \mu^{\otimes n}) = \left\{ f \in L^2(\mathcal{X}^n) : \begin{array}{l} \forall \pi \in S_n \ \forall x_1 \ldots x_n \in \mathcal{X} \\ f(x_{\pi(1)} \ldots x_{\pi(n)}) = f(x_1 \ldots x_n) \end{array} \right\}$$

Thm $\forall n \geq 1 \ \forall f \in L^2_{sym}(\mathcal{X}^n)$ simple

$$E\left(\left[\int f \, d{:}W^{\otimes n}{:} \right]^2 \right) = n! \int f^2 \, d\mu^{\otimes n}$$

so $f \longmapsto \int f \, d{:}W^{\otimes n}{:}$ extends uniquely to continuous linear map

$$L^2_{sym}(\mathcal{X}^n, \mathcal{G}^{\otimes n}, \mu^{\otimes n}) \longrightarrow L^2(\Omega, \mathcal{F}^W, P)$$

which is an isometry (modulo $n!$). Moreover,

$$\mathcal{H}_n = \left\{ \int f \, d{:}W^{\otimes n}{:} \; : \; f \in L^2_{sym}(\mathcal{X}^n, \mathcal{G}^{\otimes n}, \mu^{\otimes n}) \right\}$$

and $\forall Y \in L^2(\Omega, \mathcal{F}^W, P) \ \forall n \geq 1 \ \exists f_n \in L_{sym}(\mathcal{X}^n)$: $\; Y = EY + \sum_{n \geq 1} \int f_n \, d{:}W^{\otimes n}{:}$ converging in L^2.

Pf Let f be simple as above. Assume f symmetric and $\{A_i\}_{i=1}^m$ disjoint, non-empty:

Then $\forall \pi \in S_n$: $a_{\pi(i_1)\dots\pi(i_n)} = a_{i_1\dots i_n}$.

Now: $$E\left(\left[\int f \, d{:}W^{\otimes n}{:}\right]^2\right) = \sum_{i_1\dots i_n}\sum_{j_1\dots j_n} a_{i_1\dots i_n} a_{j_1\dots j_n} E\left({:}W(A_{i_1})\dots W(A_{i_n}){:}\;{:}W(A_{j_1})\dots W(A_{j_n}){:}\right)$$

$$= \text{———}\,''\,\text{———} \sum_{\pi\in S_n} \prod_{k=1}^n \underbrace{E\left(W(A_{i_k}) W(A_{j_{\pi(k)}})\right)}_{=\mu(A_{i_k}\cap A_{j_{\pi(k)}})}$$

$$= \sum_{\pi\in S_n} \sum_{i_1\dots i_n}\sum_{j_1\dots j_n} a_{i_1\dots i_n} a_{j_{\pi(1)}\dots j_{\pi(n)}} \prod_{k=1}^n \underbrace{\mu(A_{i_k}\cap A_{j_k})}_{=\delta_{i_k j_k}\mu(A_{i_k})}$$

$$\overset{\text{sym}}{=} n! \sum_{i_1\dots i_n} (a_{i_1\dots i_n})^2 \prod_{k=1}^n \mu(A_{i_k}) = n! \int f^2 \, d\mu^{\otimes n}. \quad \boxtimes$$

The remaining statements follow from previous reasoning.

Notes: Higher order PW integrals appear naturally:

Lemma $\forall f_1 \dots f_n \in L^2(\mathcal{X}, \mathcal{G}, \mu)$:

$$:\prod_{i=1}^n \int f_i \, dW: \; = \int f_1 \otimes \dots \otimes f_n \, d{:}W^{\otimes n}{:}$$

where $$f_1 \otimes \dots \otimes f_n (x_1 \dots x_n) := \frac{1}{n!} \sum_{\pi \in S_n} \prod_{i=1}^n f_i(x_{\pi(i)}) \quad \in L^2_{sym}(\mathcal{X}^n)$$

is symmetrized tensor product.

Wick product acts "sort of" on all L^2

$$X \in \mathcal{H}_n,\ Y \in \mathcal{H}_m \longmapsto X \odot Y := \mathrm{proj}_{\mathcal{H}_{n+m}}(XY)$$

$X = X_1 \ldots X_n + \text{lower order}$, $Y = Y_1 \ldots Y_m + \text{lower order}$ $\Rightarrow XY = X_1 \ldots X_n Y_1 \ldots Y_m + \text{lower order}$.

Lemma $\forall\, n,m \geq 1\ \exists\, c_{n,m} \in (0,\infty)\ \forall X \in \mathcal{H}_n\ \forall Y \in \mathcal{H}_m$:

$$\| X \odot Y \| \leq c_{n,m} \|X\| \, \|Y\|$$

This extends $\odot$ to $\bigcup_{n \in \mathbb{N}} \overline{\mathrm{span}\{\mathcal{H}_0 \cup \ldots \cup \mathcal{H}_n\}}$, unfortunately not to L^2

Note $\odot$ commutative, associative, distributive w.r.t. $+$

For $f(x) = \sum_{n \geq 0} a_n x^n$ entire function, we define

$$:f(X): \;= \sum_{n \geq 0} a_n :X^n: \quad \text{wherever sum converges in } L^2.$$

The convergence requires:

$$E\left([:f(X):]^2\right) = \sum_{n \geq 0} |a_n|^2 \underbrace{E(:X^n: :X^n:)}_{= n! \|X\|^{2n}}$$

So we need $\sum_{n=0}^{\infty} |a_n|^2 n! \, \|X\|^{2n} < \infty$.

Wick-ordered exponential:

Lemma $X \in \mathcal{H}_1$: $:e^{tX}: = e^{tX - \frac{1}{2}t^2 E(X^2)}$

Pf. $:e^{tX}: = \sum_{n=0}^{\infty} \frac{t^n}{n!} :X^n:$

WLOG: $E(X^2) = 1$.

n-th Hermit polynomial

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(X)$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} (-1)^n \left(e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} \right) \Big|_{x := X}$$

$$= e^{X^2/2} e^{-(X-t)^2/2} = e^{tX - t^2/2}.$$

$$:e^{\int f dw}: = \sum_{n \geq 0} \frac{1}{n!} \int f \otimes \cdots \otimes f \, :dW^{\otimes n}:$$

(time-ordered exponential)

Iterated Itô integrals. Itô discovered a representation of $L^2(\Omega, \mathcal{F}^B, P)$ by:

$$I_t^{(n)}(f) := \int_0^t \left(\int_0^{t_n} \left(\cdots \left(\int_0^{t_2} f(t_1 \ldots t_n) \, dB_{t_1} \right) dB_{t_2} \right) \cdots \right) dB_{t_n}$$

Note: We only need values of f on $D_n \cap [0,t]^n$ where

$$D_n := \{ (t_1 \ldots t_n) \in \mathbb{R} ; \; 0 \leq t_1 < t_2 < \cdots < t_n \},$$

also $I_t^{(n)}(f) = \int_0^t I^{(n-1)}(f_s) \, dB_s$ where $f_s(t_1 \ldots t_{n-1}) := f(t_1 \ldots t_{n-1}, s) \mathbb{1}_{D_n}(t_1 \ldots t_{n-1}, s)$

Thm $\forall n \geq 1\ \forall f \in L^2(D_n)\ \exists \{ I_t^{(n)}(f) : t \geq 0 \}$ continuous, adapted process s.t.

$$n = 1 \Rightarrow \forall f \in L^2(\mathbb{R}_+)\ \forall t \geq 0 : \quad I_t^{(1)}(f) = \int_0^t f(s)\, dB_s \quad \text{a.s.}$$

and $\forall n \geq 2\ \exists Y \in \mathcal{V}_B\ \forall f \in L^2(D_n)$:

$$\forall t \geq 0 : \quad Y_t = I_t^{(n-1)}(f_t) \quad \text{a.s.}$$

and

$$I_t^{(n)}(f) = \int_0^t Y_s\, dB_s \quad \text{a.s.}$$

Here, we assume $\mathcal{F}_0$ contains all P-null sets.

Pf: f simple: $\exists m \geq 1\ \exists s_0 = 0 < s_1 < \dots < s_m$:

$$f(t_1 \dots t_n) = \sum_{0 \leq j_1 < \dots < j_n \leq m} a_{j_1 \dots j_m} \prod_{k=1}^n 1_{(s_{j_k - 1}, s_{j_k}]}(t_k)$$

then set

$$I_t^{(n)}(f) := \sum_{0 \leq j_1 < \dots < j_n \leq m} a_{j_1 \dots j_m} \prod_{k=1}^n \left(B_{t \wedge s_{j_k}} - B_{t \wedge s_{j_{k-1}}} \right)$$

This constructs $I_t^{(n)}(f)$ $\forall f \in L^2(D_n \cap [0,t]^n)$

$$E\left(I_t^{(n)}(f)^2 \right) = \sum_{0 \leq j_1 < \dots < j_k \leq n} (a_{j_1 \dots j_m})^2 \prod_{k=1}^n (t \wedge s_{j_k} - t \wedge s_{j_{k-1}})$$

$$= \int_{D_n \cap [0,t]^n} f(t_1 \dots t_n)^2\, dt_1 \dots dt_n$$

<u>Issue</u>: prove continuity in t and relation to $I^{(n-1)}$