Higher order Paley-Wiener integrals

last $X = \{X_t : t \in T\}$ Gaussian process (mean zero for today)

$\mathcal{H}_0 := \{\lambda : \lambda \in \mathbb{R}\}$, $\mathcal{H}_1 = GHS(X)$

$$\mathcal{H}_{n+1} = \overline{\{Y \in \operatorname{span}\{X_{t_1} \cdots X_{t_n} : t_1 \ldots t_n \in T\} : Y \perp \mathcal{H}_0, \ldots, \mathcal{H}_n\}}^{L^2(\Omega, \mathcal{F}^X, P)}$$

Wiener chaos expansion:

$$L^2(\Omega, \mathcal{F}^X, P) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n. \quad \ldots \quad \forall Y \in L^2 : Y = \sum_{n=0}^{\infty} Y_n$$

$\forall m, n \in \mathbb{N} : m \neq n \Rightarrow \mathcal{H}_m \perp \mathcal{H}_n.$ where $\forall n \in \mathbb{N} : Y_n \in \mathcal{H}_n$.

For $X = W$ = white noise on $(\mathcal{X}, \mathcal{G}, \mu)$.

we observed: $\mathcal{H}_2 = \overline{\operatorname{span}\{W(A)W(B) - \mu(A \cap B) : A, B \in \mathcal{G}\}}^{L^2}$

An element of span$\{\ldots\}$ takes the form

$$\int f \, d{:}W \otimes W{:} := \sum a_{ij} \left[W(A_i) W(A_j) - \mu(A_i \cap A_j) \right]$$

for some f of the form

$$f = \sum_{i,j=1}^{m} a_{ij} \mathbb{1}_{A_i \times A_j}$$

wlog: $\{a_{ij}\}$ symmetric

$\{A_i\}$ disjoint

$$E\left(\left(\int f \, d{:}W\otimes W{:}\right)^2\right) = E\left(\left(2\sum_{1\le i<j\le n} a_{ij}\, W(A_i)W(A_j) + \sum_{i=1}^{n} a_{ii}[W(A_i)^2-\mu(A_i)]\right)^2\right)$$

$$= 4\sum_{i<j} a_{ij}^2\, E(\underbrace{W(A_i)^2}_{\mu(A_i)})\,E(\underbrace{W(A_j)^2}_{\mu(A_j)}) + \sum_{i=1}^{n} a_{ii}^2\, E\big(\underbrace{[W(A_i)^2-\mu(A_i)]^2}_{2\,\mu(A_i)^2}\big)$$

$$= 2\left[2\sum_{i<j} a_{ij}^2\,\mu(A_i)\mu(A_j) + \sum_{i} a_{ii}^2\,\mu(A_i)^2\right] = 2\sum_{i,j} a_{ij}^2\,\mu(A_i)\mu(A_j)$$

$$= 2\int f^2 \, d\mu\otimes\mu$$

As a consequence:

$\int f \, d{:}W\otimes W{:}$ exists $\forall f\in L^2(\mathcal{X}\times\mathcal{X}, \mathcal{G}\otimes\mathcal{G}, \mu\otimes\mu)$

and for symmetric functions $f\longmapsto \int f\, d{:}W\otimes W{:}$ is an isometry (mod 2) of $L^2_{sym}(\mathcal{X}\times\mathcal{X}, \mathcal{G}\otimes\mathcal{G}, \mu\otimes\mu)\to L^2(\Omega, \mathcal{F}^W, P)$

In particular,

$$\mathcal{H}_2 = \left\{\int f\, d{:}W\otimes W{:} \;\; f\in L^2_{sym}(\mathcal{X}\times\mathcal{X}, \mathcal{G}\otimes\mathcal{G}, \mu\otimes\mu)\right\}.$$

Wick product / normally-ordered product

Let $X = \{X_t : t \in T\}$ mean zero Gaussian process.

$\forall Y_1 \dots Y_k \in GHS(X) = \mathcal{H}_1$: $\quad :Y_1 \cdots Y_k: \ := \mathrm{proj}_{\mathcal{H}_k}(Y_1 \cdots Y_k)$.

Ex $\forall Y_1, Y_2, Y_3 \in \mathcal{H}_1$:

$$:Y_1: = Y_1$$

$$:Y_1 Y_2: = Y_1 Y_2 - \mathrm{Cov}(Y_1 Y_2)$$

$$:Y_1 Y_2 Y_3: = Y_1 Y_2 Y_3 - Y_1 \mathrm{Cov}(Y_2 Y_3) - Y_2 \mathrm{Cov}(Y_1, Y_3) - Y_3 \mathrm{Cov}(Y_1, Y_2)$$

Lemma $\forall Y_1 \dots Y_n \in \mathcal{H}_1$:

$$:Y_1 \cdots Y_n: = \sum_{\substack{A \subseteq \{1 \dots n\} \\ |A| \text{ even}}} (-1)^{|A|/2} \Big[\underbrace{\sum_{\pi \in \mathcal{P}(A)}}_{\text{pairings of } A} \prod_{(i,j) \in \pi} E(Y_i Y_j) \Big] \prod_{i \notin A} Y_i$$

Pf: Based on: $\forall Z_1 \dots Z_{2n} \in \mathcal{H}$:

$$E(Z_1 \cdots Z_{2n}) = \sum_{k=2}^{2n} E(Z_1 Z_k) E\Big(\prod_{\substack{j=1 \dots 2n \\ j \neq 1,k}} Z_j\Big)$$

$$= \underbrace{\sum_{\pi \in \mathcal{P}(\{1 \dots 2n\})}}_{\text{pairings of } \{1 \dots n\}} \prod_{(i,j) \in \pi} E(Z_i Z_j)$$

(Wick pairing formula)

NTS: $\forall k=1\ldots n-1 \quad \forall Z_1\ldots Z_k$: $E(Z_1\ldots Z_k \quad \text{RHS}) = 0$

$$E(Z_1\ldots Z_k \quad \text{RHS}) = \sum_{\substack{B\subseteq\{1\ldots n\}\\ |B| \text{ even}}} \underbrace{\sum_{\substack{A\subseteq B\\ |B\setminus A| = \text{even}}} (-1)^{|A|/2} \left[\sum_{\pi\in\mathcal{P}(B)} \prod_{(i,j)\in\pi} E(Y_iY_j)\right]}_{\substack{\shortparallel \\ 0}} \times \text{term that depends only on } B$$

Needs to be written subsets of pairings. See notes.

0 unless ~~$A=\{1\ldots 2n\}$~~

unless B is empty!

This cannot be achieved when $k<n$.

<u>Lemma</u> $\forall Z_1\ldots Z_n \ \forall Y_1\ldots Y_n$:

$$E(:Z_1\ldots Z_n: \ :Y_1\ldots Y_n:) = \sum_{\pi\in S_n} \prod_{i=1}^{n} E(Z_i Y_{\pi(i)})$$

Notice: No pairing within colons allowed!

For $f = \sum_{i_1,\ldots i_n=1}^{m} a_{i_1\ldots i_n} 1_{A_{i_1}\times\ldots\times A_{i_n}}$

we set $\int f \, d{:}W^{\otimes n}{:} := \sum_{i_1\ldots i_n=1}^{m} a_{i_1}\ldots a_{i_n} :W(A_{i_1})\cdots W(A_{i_n}):$

Additivity $A\mapsto W(A)$ + linearity of Wick product $\Rightarrow$ $\int f\, d{:}W^{\otimes n}{:}$ independent of representation.

Let $L^2_{\text{sym}}(\mathcal{X}^n, \mathcal{G}^{\otimes n}, \mu^{\otimes n}) = \{f\in L^2(\mathcal{X}^n, \mathcal{G}^{n}, \mu^{n}) : \forall\pi\in S_n \ \forall x_1\ldots x_n : f(x_{\pi(1)}\ldots x_{\pi(n)}) = f(x_1\ldots x_n)\}$

Thm $\forall f : \mathcal{X}^n \to \mathbb{R}$ symmetric & simple:

$$E\left(\left(\int f \, d{:}W^{\otimes n}{:}\right)^2\right) = n! \int f^2 \, d\mu^{\otimes n}$$

and so $f \longmapsto \int f \, d{:}W^{\otimes n}$ extends uniquely to continuous, linear map

$$L^2_{sym}(\mathcal{X}^n, \mathcal{G}^{\otimes n}, \mu^{\otimes n}) \longrightarrow L^2(\Omega, \mathcal{F}^W, P).$$

Moreover, $\mathcal{H}_n = \left\{ \int f \, d{:}W^{\otimes n}{:} \; : \; f \in L^2_{sym}(\mathcal{X}^n, \mathcal{G}^{\otimes n}, \mu^{\otimes n}) \right\}$

and

$$\forall Y \in L^2(\Omega, \mathcal{F}^W, P) \;\; \forall n \geq 1 \;\; \exists f_n \in L_{sym}(\mathcal{X}^n, \mathcal{G}^{\otimes n}, \mu^{\otimes n})$$

s.t.

$$Y = EY + \sum_{n=1}^{\infty} \int f_n \, d{:}W^{\otimes n}{:}$$