Final Remarks on Cameron-Martin

$f \in CMS(B)$ and

$\mathcal{L}aw(B+f) \ll \mathcal{L}aw(B) \iff P(B+f \in A) = E\left(1_{B\in A}\, e^{Y - \frac{1}{2}EY^2}\right)$

where $Y = \phi^{-1}(f)$.

need to check that concepts are same on $C[0,t]$ and $\mathbb{R}^{[0,t]}$.

Recall: $CMS(B) = \left\{ \int_0^t f'(s)\, dB_s : f' \in L^2([0,t]) \right\}$.

writing $f(u) = \int_0^u f'(s)\,ds$, $Y = \phi^{-1}(f) = \int_0^t f'(s)\,dB_s$.

RHS above $= E\left(1_{B\in A}\, e^{\int_0^t f'(s)dB_s - \frac{1}{2}\int_0^t f'(s)^2 ds}\right)$

Note: What about non-zero mean processes?

Two non-zero mean Gaussian processes are mutually AC $\iff$ the difference of means is CMS.

For W = white noise on $(\mathcal{X}, \mathcal{G}, \mu)$. Change mean by adding a finite measure ν.

$\mathcal{L}aw(W+\nu) \ll \mathcal{L}aw(W) \iff \nu \in CMS(W) \ldots \left(\nu \ll \mu \wedge \frac{d\nu}{d\mu} \in L^2(\mu)\right)$

and $P(W+\nu \in A) = E\left(1_{W\in A}\, e^{\int \frac{d\nu}{d\mu} dW - \frac{1}{2}\int \left(\frac{d\nu}{d\mu}\right)^2 d\mu}\right)$

Wiener chaos expansion (Wiener 1938)

Idea Represent functions of Gaussian process in form "power" series.

Lemma Let $X \overset{\text{"}}{=} \{X_t : t \in T\}$ = Gaussian process, $\mathcal{F}^X := \sigma(X_t : t \in T)$. Then

$$\forall Y \in L^\infty(\Omega, \mathcal{F}^X, P): \quad \forall n \in \mathbb{N}\ \forall t_1 \dots t_n \in T: \ E\Big(\Big(\prod_{i=1}^{n} X_{t_i}\Big) Y\Big) = 0$$

$$\Rightarrow Y = 0. \qquad (EY = 0 \text{ included})$$

Pf: Condition implies: $\forall t_1 \dots t_n \in T\ \forall \lambda_1 \dots \lambda_n \in \mathbb{R}$

$$E\left(e^{\sum_{j=1}^{n} \lambda_j X_{t_j}}\, Y\right) = 0$$

This function is analytic in $\lambda_1 \dots \lambda_n$. So we also get

$$E\left(e^{i \sum_{j=1}^{n} \lambda_j X_{t_j}}\, Y\right) = 0$$

Take $\hat{f} \in L^2(\mathbb{R}^n, \text{leb})$ and integrate against LHS with respect to Lebesgue:

So we get

$$\forall f \in L^2(\mathbb{R}^n, \mathrm{Leb}): \quad E\left(f(X_{t_1}, \dots X_{t_n}) Y\right) = 0$$

Hence: $E\left(Y \mid \sigma(X_{t_1} \dots X_{t_n})\right) = 0$

Now: let $A \in \mathcal{F}^X$. By countability curse, $A \in \sigma(X_t : t \in S)$ for some $S \subseteq T$ countable.

Lévy Forward Thm: $E\left(Y \mid \sigma(X_t : t \in S)\right) = 0$.

For $A := \{Y > 0\}$, we get $E(Y^+) = 0$. Since $E(Y) = 0$ we have $Y = 0$ a.s.

$\boxtimes$

Corollary $\bigcup_{n \in \mathbb{N}} \operatorname{span}\left\{ \prod_{i=1}^{n} X_{t_i} : t_1 \dots t_n \in T \right\}$ is dense in $L^2(\Omega, \mathcal{F}^X, P)$.

Set $\mathcal{H}_0 = \{ Y \in L^2(\Omega, \mathcal{F}^X, P) : Y = EY \}$. ... constants

and recursively:

$$\mathcal{H}_{n+1} := \left\{ Y \in \overline{\operatorname{span}\left(\prod_{i=1}^{n+1} X_{t_i} : t_1 \dots t_{n+1} \in T\right)} : Y \perp \mathcal{H}_0 \cup \dots \cup \mathcal{H}_n \right\}$$

Lemma $\forall m,n \in \mathbb{N}: m \neq n \Rightarrow \mathcal{H}_m \perp \mathcal{H}_n$

and $L^2(\Omega, \mathcal{F}^X, P) = \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$ ← - - - Limits are included

Pf Pick $Y \in L^2(\dots)$. Then Corollary: $\exists \{Z_n\}_{n\in\mathbb{N}} \in L^2$: $\forall n\in\mathbb{N}: Z_n \in \bigcup_{j=0}^{n} \operatorname{span}\left(\prod_{i=1}^{j} X_{t_i} : t_1 \dots t_j \in T\right)$ s.t. $Z_n \to Y$ in L^2.

Set $Y_n := Z_n - Z_{n-1} - \mathrm{proj}_{\mathcal{H}_{n-1}}(Z_n - Z_{n-1}) + \mathrm{proj}_{\mathcal{H}_n}(Z_{n+1} - Z_n)$. Then

$$\mathrm{proj}_{\mathcal{H}_0}(Z_1) + \sum_{k=1}^{n} Y_k = \underbrace{Z_0 + \sum_{k=1}^{n}(Z_k - Z_{k-1})}_{Z_n} + \mathrm{proj}(Z_{n+1} - Z_n) \xrightarrow{L^2} Y$$

Since $Y_n \in \mathcal{H}_n$, we're done. ☒

Note
- Above referred to as Wiener chaos expansion
- $\mathcal{H}_n$ = n-th Wiener chaos (space)
- If X zero mean, then $\mathcal{H}_1 = \mathrm{CMS}(X)$. (In general, $\mathcal{H}_1 = \mathrm{CMS}(X - EX)$)

Ex: T = singleton.

Lemma Let $X = N(0,1)$. Then $L^2(\Omega, \mathcal{F}^X, P) = \bigoplus_{n\in\mathbb{N}} \{\lambda h_n(X) : \lambda \in \mathbb{R}\}$

where $h_n(x) := e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$ is n-th Hermite polynomial.

Pf $\{h_n\}_{n\in\mathbb{N}}$ are orthogonal w.r.t. $N(0,1)$. ☒

Ex $X = W$ = white noise on $(\mathcal{X}, \mathcal{G}, \mu)$, $\mu(\mathcal{X}) < \infty$.

$\mathcal{H}_1 = \{ \int f \, dW : f \in L^2(\mu) \}$.

Q: What's $\mathcal{H}_2$?

By definition $\mathcal{H}_2$ contains elements of form $W(A)W(B) - \alpha W(C) - \beta =: Y$ that are $\perp \mathcal{H}_0, \mathcal{H}_1$

$\perp \mathcal{H}_0$: $E(Y) = 0 \Rightarrow \beta = \mu(A \cap B)$

$E(W(D)Y) = 0$ $\quad E\Big(\underbrace{[W(A)W(B) - \mu(A\cap B)]\,W(D)}_{=0 \text{ by symm.}}\Big) = \alpha E(W(C)W(D)) = \alpha \mu(C \cap D)$

$\Rightarrow \alpha = 0$

So $\mathcal{H}_2 \supseteq \{ W(A)W(B) - \mu(A \cap B) : A, B \in \mathcal{G} \}$.

Same as for Paley-Wiener, if $f = \sum_{i,j=1}^{n} \alpha_{ij} 1_{A_i \times A_j}$

then $\int f \, d{:}W \otimes W{:} := \sum_{i,j=1}^{n} \alpha_{ij} \left[W(A_i)W(A_j) - \mu(A_i \cap A_j) \right]$

Next time: $E\left(\left(\int f \, d{:}W\otimes W{:}\right)^2\right) \leq C \|f\|^2_{L^2(\mu \otimes \mu)}$