Karhunen-Loève expansion

$\{X_t : t \in T\}$ mean-zero stoch process indexed by T ($\mathbb{R}$-valued)

$$GHS(X) := \overline{\operatorname{span}\{X_t : t \in T\}}^{L^2}$$

$$CMS(X) := \{\Phi(Y) : Y \in GHS(X)\}$$

where $\phi_t(Y) := E(X_t Y)$ (autocorrelation function.)

Ex $B = \{B_s : s \le t\}$

$$GHS(B) = \left\{ \int_0^t f(s)\,dB_s : f \in L^2[0,t] \right\}. \qquad \Phi_u\left(\int_0^t f(s)\,dB_s\right) = \int_0^u f(s)\,ds$$

$$CMS(B) = \left\{ \int_0^{\cdot} f(s)\,ds : f \in L^2[0,t] \right\}$$

$$= \left\{ F \in AC[0,t] : F(0) = 0 \wedge F' \in L^2[0,t] \right\}.$$

$$\left\langle \int_0^{\cdot} f(s)\,ds, \int_0^{\cdot} g(s)\,ds \right\rangle = E\left(\Phi^{-1}\left(\int_0^{\cdot} f(s)\,ds\right), \Phi^{-1}\left(\int_0^{\cdot} g(s)\,ds\right)\right) = E\left(\left(\int_0^t f(s)\,dB_s\right)\left(\int_0^t g(s)\,dB_s\right)\right) = \int_0^t f(s)g(s)\,ds,$$

Note $t \mapsto X_t$ is cont. in $L^2 \Rightarrow CMS(X) \subseteq C(T)$.

Lemma: Let $\{e_\alpha\}_{\alpha\in I}$ be an ONB in $CMS(X)$. Set $Z_\alpha := \phi^{-1}(e_\alpha)$. Then:

1) $\{Z_\alpha : \alpha \in I\}$ are iid $N(0,1)$ (Gaussian key).

2) $\forall t\in T$: $\sum_{\alpha\in I} e_\alpha(t)^2 < \infty$

3) $\forall t \in T$: $X_t = \sum_{\alpha\in I} e_\alpha(t) Z_\alpha$ converges a.s. & L^2

requires ordering of I

Pf ϕ isometry $\Rightarrow E(Z_\alpha Z_\beta) = \langle e_\alpha, e_\beta\rangle = \delta_{\alpha\beta}$. $\{Z_\alpha\}$ Gaussian $\Rightarrow$ independent.

For any $Y \in GHS$:

$$\sum_{\alpha\in I} E(YZ_\alpha)^2 < \infty$$ by Parseval. & $$Y = \sum_{\alpha\in I} E(YZ_\alpha) Z_\alpha,$$

$\{Z_\alpha : \alpha\in I\}$ is ONB in $GHS(X)$

where sum converges in L^2.

For $Y = X_t$, 2) + 3-series thm $\Rightarrow$ converge also a.s. $\boxed{X}$

For $B = \{B_u : u \le t\}$, the above gives:

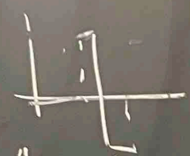
$$B_u = \sum_{k=1}^{\infty} \left(\int_0^u f_k(s)\,ds \right) Z_k$$

$\{f_k\}$ = ONB in $L^2[0,t]$

Z_k = iid $\mathcal{N}(0,1)$.

This is Lévy construction of SBM!

f_0 = constant

f_k = 

Haar basis in $L^2[0,1]$, and $\left\{ \int_0^{\cdot} f_k(s)\,ds \right\}$ is Schauder basis in CMS(B).

Q: Is uniformity of convergence special to choice of basis?

Lemma Suppose T is compact and X is continuous (pathwise). Then each element $\{e_\alpha\}$ of ONB in CMS(X) is bounded, the basis is at most countable and

$$\sup_{t \in T} \sum_{\alpha \in I} e_\alpha(t)^2 < \infty.$$

In fact, enumerating the basis as $\{e_{\alpha_k}\}_{k \in \mathbb{N}}$:

$$\sup_{t \in T} \sum_{k \ge n} e_{\alpha_k}(t)^2 \xrightarrow[n \to \infty]{} 0.$$

Pf: X cont & T compact $\Rightarrow$ $GHS(X)$ is separable. $\Rightarrow$ ONB at most countable.

Since also $t \longmapsto E(X_t^2)$ is continuous, $\Rightarrow$ $\sup_{t\in T} E(X_t^2) < \infty$ by compactness of T.

Parseval $E(X_t^2) = \sum_\alpha e_\alpha(t)^2$ is uniformly bounded. $\boxtimes$

Denote $\varphi_n(t) = \sum_{k\geq n} e_{\alpha_k}(t)^2$. Then $\varphi_n \downarrow 0$ pointwise, φ_n continuous $\forall n\geq 0$,

$\Rightarrow \varphi_n \downarrow 0$ uniformly by Dini's Thm.

$\boxtimes$

Thm Let $\{X_t : t\in T\}$ be continuous with T compact. Then for any ONB $\{e_n : n\in\mathbb{N}\}$ in $CMS(X)$ and $Z_n := \phi^{-1}(e_n)$:

$$\lim_{n\to\infty} \sup_{t\in T} \left| X_t - \sum_{k=1}^{n} e_k(t) Z_k \right| = 0. \quad \text{a.s.}$$

In short, the convergence $X_t = \sum_{k=1}^{\infty} e_k(t) Z_k$ is uniform in T. a.s.

Note Convergence we had for $\sup_{t\in T} \sum_{\alpha\in I} e_\alpha(t)^2 < \infty$ is uniformity in L^2.

So we get $X_t = \sum_{\alpha \in I} e_\alpha(t) Z_\alpha$ and $\{Z_\alpha\}$ are not a basis

what we are doing is to embed CMS(X) into $\mathcal{B} = C(T)$

and claim convergence in norm of $\mathcal{B}$.

Thm (Itô-Nishio Thm) Let $\mathcal{B}$ = Banach space, $\{X_n\}_{n=0}^\infty$ independent $\mathcal{B}$-valued RV's and $X_n \overset{d}{=} -X_n$. TFAE.

1) $\sum_{k=0}^{\infty} X_k$ converges in norm a.s.

2) $\forall \varphi \in \mathcal{B}^*$: $\varphi\left(\sum_{k=0}^{n} X_k\right)$ converges in law.

Lévy thm: $\{X_k\}_{k=0}^\infty$ are $\mathbb{R}$-val'd independent, $S_n = \sum_{k=0}^n X_k$. The TFAE:

1) S_n conv. a.s

2) S_n conv. in prob.

3) S_n conv. in law

Pf of Thm: By Itô-Nishio, NTS: $\varphi\left(\sum_{k=n}^{m} e_k Z_k\right) \xrightarrow[m,n\to\infty]{\text{law}} 0 \quad \forall \varphi \in C(T)^*$.

Riesz-Repr. thm: $\forall \varphi \in C(T)^*\ \exists\ \mu =$ finite signed measure:

$$\forall f \in C(T):\ \varphi(f) = \int_T f\, d\mu$$

So $E\left|\varphi\left(\sum_{k=n}^{m} e_k Z_k\right)\right| \leq \int_T E\left|\sum_{k=n}^{m} e_k(\cdot) Z_k\right| d|\mu| \underset{CS}{\leq} \int_T \sqrt{\sum_{k=n}^{m} e_k(t)^2}\ |\mu|(dt)$

By unifom converge of $\sum_{k=1}^{n} e_k(\cdot)^2$, RHS $\to 0$ as $m \to \infty$ and $n \to \infty$.

Corollary For any ONB $\{f_k\}$ in $L^2[0,t]$, $\{Z_k\}_{k\in\mathbb{N}}$ iid $\mathcal{N}(0,1)$:

The series $\sum_{k=1}^{\infty} \left(\int_0 f_k(s)\, ds\right) Z_k$ converges uniformly a.s.

and defines a SBM.