Lemma Let $f : \mathbb{R}_+ \to \mathbb{R}$ be s.t. $f(0) = 0$. Then $\forall t > 0$:

$$P(B + f \in \cdot)|_{\mathcal{F}_t} \ll P(B \in \cdot)|_{\mathcal{F}_t} \Rightarrow f \in AC[0,t] \wedge f' \in L^2([0,t]).$$

Pf .... (last time) $\Pi = \{0 = t_0 < \dots < t_m = t\}$, define

$$g_\Pi(s) := \sum_{i=1}^{m} \frac{f(t_i) - f(t_{i-1})}{t_i - t_{i-1}} 1_{(t_{i-1}, t_i]}(s)$$

For $\Pi = D_n = \{t i 2^{-n} : i = 0, \dots, 2^n\}$, set $g_n := g_{D_n}$.

We showed that assumptions imply $\sup_{n \geq 1} \int_0^t g_n(s)^2 ds < \infty$.

claim $f \in AC[0,t]$

Pf Let $[a_1, b_1] \dots [a_k, b_k]$ be disjoint intervals with endpoints in D_n for some $n \geq 1$.

Then $h(s) = \begin{cases} \text{sgn}(f(b_i) - f(a_i)) & \text{on } [a_i, b_i] \quad (i = 1 \dots k). \\ 0 & \text{else} \end{cases}$

$$\sum_{i=1}^{k} |f(b_i) - f(a_i)| = \int h(s) g_n(s) ds \leq \left(\int_0^t g_n(s)^2 ds\right)^{1/2} \left(\sum_{i=1}^{k} |b_i - a_i|\right)^{1/2}$$

which implies $f \in AC[0,t]$.

So we know $f(s) = \int_0^s f'(u)\,du$.

<u>claim</u> $f' \in L^2([0,t])$.

<u>Pf</u>: Let U = uniform on $[0,t]$. Set $\mathcal{G}_n = \sigma\left(\left\{ \{s \le U \le t\} : s,t \in D_n,\ s<t \right\}\right)$

Then $g_n(U) = E(f'(U) \mid \mathcal{G}_n)$.

Levy Forward Thm: $g_n(U) \longrightarrow E\left(f'(U) \mid \sigma\left(\bigcup_{n\ge 1} \mathcal{G}_n\right)\right) = f'(U)$ a.s. ($\sigma\left(\bigcup_{n\ge 1} \mathcal{G}_n\right) = \sigma(U)$)

Fatou: $E(f'(U)^2) \le \liminf_{n\to\infty} E(g_n(U)^2) \le \sup_{n\ge 1} \frac{1}{t}\int_0^t g_n(s)^2\,ds < \infty$.

$\boxtimes$

We've proved:

<u>Thm</u> (Cameron-Martin '44) Let B = SBM on Wiener's space and $t>0$.
Then $\forall f : \mathbb{R}_+ \to \mathbb{R}$,

$$f \in AC[0,t] \wedge f(0)=0 \wedge f' \in L^2[0,t]$$

is equivalent to

$$P(B+f\in\cdot)\big|_{\mathcal{F}_t} \ll P(B\in\cdot)\big|_{\mathcal{F}_t} \wedge \frac{dP(B+f\in\cdot)\big|_{\mathcal{F}_t}}{dP(B\in\cdot)\big|_{\mathcal{F}_t}} = \exp\left\{\int_0^t f'(s)\,dB_s - \frac{1}{2}\int_0^t f'(s)^2\,ds\right\}$$

Ex Let $(\mathcal{X}, \mathcal{G}, \mu)$ = finite measure space, W := white noise associated with $(\mathcal{X}, \mathcal{G}, \mu)$.

i.e., $\{W(A) : A \in \mathcal{G}\}$ mean-zero Gaussian s.t. $\forall A, B \in \mathcal{G}$: $E(W(A)W(B)) = \mu(A \cap B)$

$$\boxed{\text{Lemma Then } GHS(W) = \left\{ \int f \, dW : f \in L^2(\mu) \right\}.}$$

Pf: same.

Representation of $GHS(X)$ as a space of functions on T

Lemma Let $\{X_t : t \in T\}$ = mean-zero Gaussian. For $Y \in GHS(X)$ and $t \in T$ set

$$\phi_t(Y) = E(Y X_t).$$

Then $\phi(Y)$ is a map $T \to \mathbb{R}$ is s.t.

$$\forall Y, \tilde{Y} \in GHS(X) : \quad \phi(Y) = \phi(\tilde{Y}) \Rightarrow Y = \tilde{Y},$$

$\Leftrightarrow \forall t \in T : \phi_t(Y) = \phi_t(\tilde{Y})$

In short: $Y \mapsto \phi(Y)$ is injective.

Pf Suppose $\forall t \in T$: $\phi_t(Y) = \phi_t(\tilde{Y})$.

Then $\forall t \in T$: $E(X_t(Y - \tilde{Y})) = 0$. Using linear comb & L^2-closure:

$$\forall Z \in GHS(X): \quad E(Z(Y - \tilde{Y})) = 0.$$

Take $Z = Y - \tilde{Y}$ to get $E((Y - \tilde{Y})^2) = 0$. $\boxtimes$

Def The set $CMS(X) := \{ \phi(Y) : Y \in GHS(X) \}$

is called the Cameron-Martin space associated with X.

Lemma
- $CMS(X)$ is linear vector space.
- $\phi : GHS(X) \to CMS(X)$ is injective
- Setting $\langle f, g \rangle := E(\phi^{-1}(f)\, \phi^{-1}(g))$ $\quad f, g \in CMS(X)$ defines inner product on $CMS(X)$ s.t. ϕ is an isometric bijection.
- $CMS(X)$ is a Hilbert space.

Lemma Let $B = SBM$ on $[0,t]$. Then

$$CMS(B) = \{ f \in AC[0,t] : f(0) = 0 \wedge f' \in L^2([0,t]) \}$$

Pf: Recall $GHS(B) = \{ \int_0^t h(s)\,dB_s : h \in L^2[0,t] \}$.

So $\phi_u\left(\int_0^t h(s)\,dB_s\right) = E\left(B_u \int_0^t h(s)\,dB_s\right) = E\left(\left(\int_0^t 1_{[0,u]}(s)\,dB_s\right)\left(\int_0^t h(s)\,dB_s\right)\right)$

$$\overset{\text{Itô}}{=} \int_0^t 1_{[0,u]}(s)\,h(s)\,ds = \int_0^u h(s)\,ds.$$

So $u \mapsto \phi_u\left(\int_0^t h(s)\,dB_s\right) \in AC[0,t]$, vanishes at 0, with derivative in $L^2[0,t]$. $\boxtimes$

Lemma W = white noise on $(\mathcal{X}, \mathcal{G}, \mu)$, $\mu(\mathcal{X}) < \infty$. Then

$$CMS(W) = \{ \nu = \text{signed measure on } (\mathcal{X}, \mathcal{G}) : \nu \ll \mu \wedge \tfrac{d\nu}{d\mu} \in L^2(\mu) \}.$$

Pf: $\phi_A\left(\int f\,dW\right) = E\left(W(A)\int f\,dW\right) = \int_A f\,d\mu$ = signed measure. $\boxtimes$

Lemma Assume T is endowed with metric s.t. $t \mapsto X_t$ is continuous in L^2.
Then $CMS(X) \subseteq C(T)$

Pf: Assume $t_n \to t$ in T. Then $\forall Y \in GMS(X)$:

$$\phi_{t_n}(Y) = E(X_{t_n} Y) \xrightarrow[n\to\infty]{} E(X_t Y) = \phi_t(Y).$$