Shifted Brownian motion

Q: Let $B = SBM$ with $B_0 = 0$. For what $f : \mathbb{R}_+ \to \mathbb{R}$ does $B+f$ look "same" as B?
Mathematically: When $\text{law}(B+f) \ll \text{law}(B)$?

Necessary conditions: $f(0) = 0$
$f \in C(\mathbb{R}_+)$

Sufficient cond?

Lemma: Let $B = SBM$ with $B_0 = 0$, let $t > 0$. Then $\forall f : \mathbb{R}_+ \to \mathbb{R}$:

f AC on $[0,t] \wedge \int_0^t [f'(s)]^2 \, ds < \infty \Rightarrow$

1) $P(B+f \in \cdot)\big|_{\mathcal{F}_t^B} \ll P(B \in \cdot)\big|_{\mathcal{F}_t^B}$

2) $$\frac{dP(B+f \in \cdot)\big|_{\mathcal{F}_t^B}}{dP(B \in \cdot)\big|_{\mathcal{F}_t^B}} = \exp\left\{ \int_0^t f'(s)\, dB_s - \frac{1}{2} \int_0^t f'(s)^2 \, ds \right\} \quad (*)$$

a.s.

$\in C([0,t])$

(Assume SBM realized on Wiener space)

Pf: Let M_t = RHS of $(*)$. Then Girsanov: $E(M_t)=1$ so $\tilde{P}(A) := E(1_A M_t)$ defines prob. measure on $(\Omega, \mathcal{F}_t^B)$. Then $\forall A \in \mathcal{F}_t^B$:

$$\tilde{P}(B - \underbrace{\int_0^\cdot f'(s)\,ds}_{f} \in A) = P(B \in A)$$

Since f deterministic, this translates into:

$$\tilde{P}(B \in A) = P(B + f \in A)$$

So we get: $P(B+f \in A) = E(1_{B \in A} M_t)$ so $\begin{cases} P(B+f\in \cdot) \ll P(B \in \cdot) \text{ on } \mathcal{F}_t^B \\ \dfrac{dP(B+f\in\cdot)}{dP(B\in\cdot)} = M_t \text{ on } \mathcal{F}_t^B. \end{cases}$

$\boxtimes$

Lemma Let $f: \mathbb{R}_+ \to \mathbb{R}$ be s.t. $f(0)=0$ and let $t>0$. Then

$$P(B+f\in\cdot)|_{\mathcal{F}_t^B} \ll P(B\in\cdot)|_{\mathcal{F}_t^B} \Rightarrow f \text{ is AC on } [0,t] \wedge \int_0^t [f'(s)]^2 ds < \infty.$$

Pf Assume laws are AC and denote

$$F := \frac{dP(B+f\in\cdot)|_{\mathcal{F}_t^B}}{dP(B\in\cdot)|_{\mathcal{F}_t^B}}.$$

Then $\forall A \in \mathcal{F}_t^B$: $P(B+f\in A) = E(1_{B\in A} F)$.

For $0 = t_0 < t_1 < \dots < t_n = t$

$$\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\frac{(B_{t_i}-B_{t_{i-1}}-(f(t_i)-f(t_{i-1})))^2}{t_i-t_{i-1}}\right\}$$

$$= \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\frac{(B_{t_i}-B_{t_{i-1}})^2}{t_i-t_{i-1}}\right\}$$

$$\times \exp\left\{+\sum_{i=1}^{n}\frac{(B_{t_i}-B_{t_{i-1}})(f(t_i)-f(t_{i-1}))}{(t_i-t_{i-1})} - \frac{1}{2}\sum\frac{(f(t_i)-f(t_{i-1}))^2}{(t_i-t_{i-1})}\right\}$$

So we get $E(F \mid \sigma(B_{t_1} \dots B_{t_n})) \overset{a.s.}{=} \exp\left\{\int_0^t g(s)\,dB_s - \frac{1}{2}\int_0^t g(s)^2\,ds\right\}$

where $g(s) = \sum_{i=1}^{n}\frac{f(t_i)-f(t_{i-1})}{t_i-t_{i-1}}\mathbb{1}_{(t_{i-1},t_i]}(s)$.

Now let $D_n := \{2^{-n}jt : j = 0, \dots, 2^n\}$, and let g_n be g for these points.

So

$$E(F \mid \sigma(B_u : u \in D_n)) = \exp\left\{\int_0^t g_n(s)\,dB_s - \frac{1}{2}\int_0^t g_n(s)^2\,ds\right\}$$

Since $\mathcal{F}_t^B = \sigma\left(\bigcup_{n\in\mathbb{N}} \sigma(B_s : s\in D_n)\right)$, Levy Forward Thm gives:

$$E(F \mid \sigma(B_s : s\in D_n)) \xrightarrow[n\to\infty]{} F \quad \text{a.s.}$$

So $\exp\left\{ \int_0^t g_n(s)\,dB_s - \frac{1}{2}\int_0^t g_n(s)^2\,ds \right\} \xrightarrow[n\to\infty]{} F(\omega)$ for a.e. B.

Absolute continuity of the laws implies it converges also for $B+f$ insted of B.

So $\exp\left\{ \int_0^t g_n(s)\,d(B+f)_s - \frac{1}{2}\int_0^t g_n(s)^2\,ds \right\} \xrightarrow[n\to\infty]{} F(\omega+f)$ a.s.

$$= \exp\left\{ \int_0^t g_n(s)\,dB_s + \frac{1}{2}\int_0^t g_n(s)^2\,ds \right\}.$$

Using Brownian reflection we conclude

$$\sup_{n\geq 0} \int_0^t g_n(s)^2\,ds < \infty.$$

<u>claim</u>: f is AC on $[0,t]$.

<u>Pf</u> Let $[a_1,b_1] \ldots [a_k,b_k]$ be disjoint intervals with endpoints in D_n.

TBC.