Continuous martingales

A bit of review:

- stochastic process $\{X_t : t \geq 0\}$, joint measurability

  progressive measurability $\{(\omega, s) \in \Omega \times [0,t] : X_s(\omega) \in A\} \in \overbrace{\mathcal{F}_t}^{\text{filtration}} \otimes \mathcal{B}(\mathbb{R}_+)$ $\quad \forall A \in \mathcal{B}(\mathbb{R}_+)$

- filtration, adaptedness $\left(\mathcal{F}_t^X := \sigma(X_s : s \leq t)\right)$

- martingale $\forall t \geq 0 : M_t \in L^1$ (adapted) $\wedge \ \forall s \leq t : E(M_t \mid \mathcal{F}_s) = M_s$ a.s.

  $\geq$ (sub)

  $\leq$ (super)

  càdlàg

  RCLL

- martingale convergence

Lemma Let $\{M_t : t \geq 0\}$ be a (sub) martingale on $(\Omega, \mathcal{F}, P)$.

Then $\exists\, \Omega^* \in \mathcal{F}$, $P(\Omega^*) = 1$ s.t. $\forall \omega \in \Omega^*$:

$\forall t \geq 0$: $\lim_{\substack{s \downarrow t \\ s \in \mathbb{Q}}} M_s$ exists

$\forall t > 0$: $\lim_{\substack{s \uparrow t \\ s \in \mathbb{Q}}} M_s$ exists

(in $\mathbb{R}$)

Moreover, if M is a martingale, filtration is RC, then $\forall t \geq 0$: $M_t = \lim_{\substack{s \downarrow t \\ s \in \mathbb{Q}}} M_s$ a.s.

Pf $E\left(U^{\mathbb{Q}}_{[0,t]}[a,b]\right) \leq \dfrac{E(M_t - a)^+}{b-a}$

RC Lévy backward thm

— inequalities

Lemma Let X be RC submartingale. Then

$$\forall t>0\ \forall \lambda>0: \quad P\Big(\sup_{s\le t} X_s > \lambda\Big) \le \frac{1}{\lambda} E\Big(X_t^+ \mathbb{1}_{\sup_{s\le t} X_s > \lambda}\Big)$$

Corollary: Let M be RC martingale. Then $\forall t>0$:

$$\forall \lambda>0\ \forall p\ge 1: \quad P\Big(\sup_{s\le t} |M_s| > \lambda\Big) \le \frac{1}{\lambda^p} E(|M_t|^p)$$

$$\forall p>1: \quad E\Big(\big(\sup_{s\le t}|M_s|\big)^p\Big) \le \Big(\frac{p}{p-1}\Big)^p E(|M_t|^p)$$

— stopping times $\quad \forall t\ge 0: \{T\le t\}\in\mathcal{F}_t$. $\qquad \{T<t\}\in\mathcal{F}_t$ .. optional time

$$\mathcal{F}_T := \big\{A\in\mathcal{F} : \big(\forall t\ge 0: A\cap\{T\le t\}\in\mathcal{F}_t\big)\big\}$$

X stoch. process, adapted $\wedge$ T - stopping time

Q: When is X_T $\mathcal{F}_T$ - measurable?

A: X progressively measurable.

Uniqueness follows from:

Lemma Let M be a continuous <u>local</u> martingale s.t. $V_t^{(1)}(M) < \infty$ for some $t > 0$. Then $P(\forall s \le t : M_s = M_0) = 1$

Pf Stop the martingale to ensure bddness & $V_t^{(1)}(M) \le K$.

$$E((M_t - M_0)^2) = E(\bar{V}_t^{(2)}(M, \Pi)) \le E\Big(\mathrm{osc}_M([0,t], \|\Pi\|) \underbrace{\bar{V}_t^{(1)}(M, \Pi)}_{\le K}\Big) \xrightarrow[\|\Pi\| \to 0]{} 0$$

$\boxtimes$

A: X progressively measurable.