9. Applications of Girsanov's theorem

We will now proceed deeper into the subject of Girsanov's theorem by first discussing the conditions under which it can be used and then giving some standard applications.

9.1 Conditions for M being a martingale.

The requirement that the local martingale M_t has unit expectation for all $t \geqslant 0$ and thus, by Lemma 8.5, constitutes a proper martingale is not at all trivial. To demonstrate this, consider the following illustrative example:

Lemma 9.1 *Let $d > 2$ and let X_t be d-dimensional Bessel process solving the SDE*

$$\mathrm{d}X_t = \frac{d-1}{2X_t}\,\mathrm{d}t + \mathrm{d}B_t \tag{9.1}$$

with initial value $X_0 = 1$. Then $M_t := X_t^{2-d}$ takes the form

$$M_t = \exp\left\{-(d-2)\int_0^t \frac{1}{X_s}\mathrm{d}B_s - \frac{(d-2)^2}{2}\int_0^t \frac{1}{X_s^2}\,\mathrm{d}s\right\} \tag{9.2}$$

which is well defined for all $t \geqslant 0$ as $\inf_{t\geqslant 0} X_t > 0$ a.s., and is thus a local martingale. Yet

$$\forall t > 0\colon \quad EM_t < 1. \tag{9.3}$$

Proof. Suppose, by way of contradiction, that $EM_T = 1$ for some $T > 0$. Since $d > 2$ implies $P(\tau_0 < \infty) = 1$, also

$$M_t' := \exp\left\{-(d-2)\int_0^t \frac{1}{X_s}1_{\{\tau_0>s\}}\mathrm{d}B_s - \frac{(d-2)^2}{2}\int_0^t \frac{1}{X_s^2}1_{\{\tau_0>s\}}\,\mathrm{d}s\right\} \tag{9.4}$$

obeys $E(M_T') = 1$. Then Theorem 8.4 shows that the process $\{\widetilde{B}_t\colon t \leqslant T\}$, for

$$\widetilde{B}_t := B_t + \int_0^t \frac{d-2}{X_s}1_{\{\tau_0>s\}}\mathrm{d}s \tag{9.5}$$

is a standard Brownian motion under the measure $\widetilde{P}(A) := E(1_A M_T')$. As this gives $\mathrm{d}B_t = \mathrm{d}\widetilde{B}_t - \frac{d-2}{X_t}\mathrm{d}t$, the SDE (9.1) transforms into

$$\mathrm{d}X_t = \Big(\frac{d-1}{2X_t} - \frac{d-2}{X_t}\Big)1_{\{\tau_0>t\}}\mathrm{d}t + \mathrm{d}\widetilde{B}_t = \frac{3-d}{2X_t}1_{\{\tau_0>t\}}\mathrm{d}t + \mathrm{d}\widetilde{B}_t. \tag{9.6}$$

Setting $d'-1 := 3-d$ gives $d' = 4-d$ and so we get that $\{X_t\colon t \leqslant T\}$ is a $(4-d)$-dimensional Bessel process under $\widetilde{P}$. As discussed in 275D, the solution is unique up to the first hitting time of zero and since the process continues along the path of $\widetilde{B}_t - \widetilde{B}_{\tau_0}$ thereafter, it is unique for all $t \geqslant 0$.

The key point for the present proof is that, since $4-d < 2$ for $d > 2$, the $(4-d)$-dimensional Bessel process does hit zero in time interval $[0,T]$ with positive probability. (We leave a proof of this to homework.) This means

$$\widetilde{P}\Big(\inf_{t\in[0,T]} X_t = 0\Big) > 0 \tag{9.7}$$

yet the assumption that $d > 2$ gives

$$P\Big(\inf_{t\in[0,T]} X_t = 0\Big) = 0 \tag{9.8}$$

in contradiction with $\widetilde{P} \ll P$ on $\mathcal{F}_T$. It follows that $E(M_T) < 1$ after all. □

An interesting additional twist of the above example is that, for any $d > 2$, the local martingale $\{M_t : t \geqslant 0\}$ in (9.2) admits moments slightly larger than one and is thus uniformly integrable. (Yet, as discussed after Theorem 1.4 and witnessed by the statement of the lemma, this is not sufficient to turn it into a proper martingale.) We leave proofs of these facts to homework.

In light of above observations, one is thus interested in natural, and reasonably sharp, conditions on a local martingale M with $M_0 = 0$ that would guarantee $E(\mathrm{e}^{M_t - \frac12\langle M\rangle_t}) = 1$. Here is one that is fairly easy to prove:

Lemma 9.2 *Let $M \in \mathscr{M}_{\mathrm{loc}}^{\mathrm{cont}}$ be such that $M_0 = 0$ and let $t > 0$. Then*

$$\exists \epsilon > 0\colon\;\; E\big(\mathrm{e}^{(\frac12+\epsilon)\langle M\rangle_t}\big) < \infty \tag{9.9}$$

implies

$$E\big(\mathrm{e}^{M_t - \frac12\langle M\rangle_t}\big) = 1. \tag{9.10}$$

Proof. Abbreviate $X_t := \mathrm{e}^{M_t - \frac12\langle M\rangle_t}$. We proceed by a localization argument. For $n \geqslant 1$, denote $\tau_n := \inf\{t \geqslant 0 : M_t \geqslant n\}$ and pick $\lambda > 1$. Pick $p > 1$ and let q be such that $p^{-1} + q^{-1} = 1$. Then some rewrites and the Hölder inequality give

$$\begin{aligned} E\big(X_{\tau_n\wedge t}^{\lambda}\big) &= E\big(\mathrm{e}^{\lambda M_{\tau_n\wedge t} - \frac{\lambda}{2}\langle M\rangle_{\tau_n\wedge t}}\big) \\ &= E\big(\mathrm{e}^{\lambda M_{\tau_n\wedge t} - \frac{p\lambda^2}{2}\langle M\rangle_{\tau_n\wedge t}}\mathrm{e}^{\frac{\lambda}{2}(p\lambda-1)\langle X\rangle_{\tau_n\wedge t}}\big) \\ &\leqslant \Big(E\big(\mathrm{e}^{p\lambda M_{\tau_n\wedge t} - \frac{p^2\lambda^2}{2}\langle M\rangle_{\tau_n\wedge t}}\big)\Big)^{1/p}\Big(E\big(\mathrm{e}^{\frac{\lambda}{2}(p\lambda-1)q\langle M\rangle_{\tau_n\wedge t}}\big)\Big)^{1/q} \end{aligned} \tag{9.11}$$

Since $\{\mathrm{e}^{p\lambda M_{\tau_n\wedge t} - \frac{p^2\lambda^2}{2}\langle M\rangle_{\tau_n\wedge t}} : t \geqslant 0\}$ is a bounded local martingale, and thus martingale, the first expectation on the right equals one. The monotonicity of $t \mapsto \langle M\rangle_t$ then gives

$$\sup_{n\geqslant1} E\big(X_{\tau_n\wedge t}^{\lambda}\big) \leqslant \Big(E\big(\mathrm{e}^{\frac{\lambda}{2}(p\lambda-1)q\langle M\rangle_t}\big)\Big)^{1/q} \tag{9.12}$$

Noting that

$$\lambda(p\lambda-1)q = p\lambda\frac{p\lambda-1}{p-1} \tag{9.13}$$

tends to one as $\lambda \downarrow 1$ and $p \downarrow 1$, we can choose $\lambda > 1$ and $p > 1$ so that the coefficient multiplying $\langle M\rangle_t$ in (9.12) is less than $\frac12 + \epsilon$. The condition (9.9) then gives that $\{X_{\tau_n\wedge t} : n \geqslant 1\}$ is bounded uniformly in L^λ and is thus uniformly integrable. In light of $E(X_{\tau_n\wedge t}) = 1$ and $X_{\tau_n\wedge t} \to X_t$ a.s., this yields $EX_t = 1$. □

The integrability condition (9.9) is actually not far from optimal. First, as examples show, the conclusion does not hold in general if (9.9) with $\epsilon < 0$ is assumed. Moreover, the case $\epsilon = 0$ is included in the positive conclusion, albeit with a different (and somewhat complicated) proof:

Theorem 9.3 (Novikov's condition) *Let $M \in \mathscr{M}_{\text{loc}}^{\text{cont}}$ obey $M_0 = 0$ and $t \geqslant 0$ be such that*

$$E\bigl(\mathrm{e}^{\frac12\langle M\rangle_t}\bigr) < \infty. \tag{9.14}$$

Then

$$E\bigl(\mathrm{e}^{M_t - \frac12\langle M\rangle_t}\bigr) = 1. \tag{9.15}$$

The condition (9.14) can further be weakened by requiring the expectation to be finite with t replaced by t', for any $t' < t$. This is known as the *Kazamaki condition*. The Kazamaki condition (and thus also Novikov's condition) is not necessary.

The main point why Novikov's condition is so well known is that it is easy to state and verify (various sufficient conditions exist for Y of the form $Y_s = a(B_s)$ that imply it). Looking for sharper conditions may in fact be somewhat beyond the point because the closer to optimal, the harder the condition will likely be to verified.

9.2 Removal of drift from Langevin-type SDE.

We now move to applications of Girsanov's theorem. The first one is on solving a particular class of Langevin-type SDEs:

Theorem 9.4 (Removal of drift term) *Suppose $a\colon [0,\infty)\times\mathbb{R}\to\mathbb{R}$ is Borel measurable and such that, for B a standard Brownian motion started from zero and $x \in \mathbb{R}$,*

$$\forall t \geqslant 0\colon\quad E\biggl(\exp\Bigl\{\frac12\int_0^t a(s,x+B_s)^2\mathrm{d}s\Bigr\}\biggr) < \infty \tag{9.16}$$

Then the SDE

$$\mathrm{d}X_t = a(t,X_t)\mathrm{d}t + \mathrm{d}B_t \tag{9.17}$$

admits a weak solution with $X_0 = x$.

Proof. Consider the setting of the standard Wiener space $(\Omega,\mathcal{F},P)$, where $\Omega := C[0,\infty)$, $\mathcal{F} := \mathcal{B}(C[0,\infty))$ and P is the Wiener measure. The Brownian motion is then realized by coordinate projections, $B_t(\omega) := \omega(t)$, and the σ-algebra $\mathcal{F}_t^B$ can be identified with $\mathcal{B}(C[0,t])$. Given $T > 0$, let $\widetilde{P}_T$ be the measure on$(\Omega,\mathcal{F}_T^B)$ defined by $\widetilde{P}_T(A) := E(1_A M_T)$, where E is expectation with respect to P and

$$M_t := \exp\biggl\{\int_0^t a(s,x+B_s)\mathrm{d}s - \frac12\int_0^t a(s,x+B_s)^2\mathrm{d}s\biggr\}. \tag{9.18}$$

By Theorem 9.3, the condition (9.16) gives that $\widetilde{P}_T$ is a probability measure. The fact that M is a martingale in turn shows

$$\forall s \leqslant t\,\forall A \in \mathcal{F}_s^B\colon\quad \widetilde{P}_t(A) = \widetilde{P}_s(A) \tag{9.19}$$

which means that the measure spaces $\{(\Omega, \mathcal{F}_t^B, \widetilde{P}_t)\colon t \geqslant 0\}$ form a Komogorov-consistent family. As the underlying measurable space is standard Borel and $\sigma(\bigcup_{t\geqslant 0} \mathcal{F}_t^B) = \mathcal{F}$, the Kolmogorov Extension Theorem gives that $\widetilde{P}_t$ is the restriction to $\mathcal{F}_t^B$ of a unique probability measure $\widetilde{P}$ on $(\Omega, \mathcal{F})$.

Thanks to the fact that $\widetilde{P}|_{\mathcal{F}_T^B} = \widetilde{P}_T$ for each $T > 0$, Girsanov's Theorem gives that, under $\widetilde{P}$, the process $\{\widetilde{B}_t\colon t \in [0, T]\}$ where

$$\widetilde{B}_t := B_t - \int_0^t a(s, x + B_s)\mathrm{d}s \tag{9.20}$$

has the law of a standard Brownian motion on $[0, T]$. As this determines the law on $(\Omega, \mathcal{F})$, the process $\{\widetilde{B}_t\colon t \geqslant 0\}$ is a standard Brownian motion under $\widetilde{P}$. Using the shorthand $X_t := x + B_t$, we can rewrite this as

$$X_t = x + \int_0^t a(s, X_s)\mathrm{d}t + \widetilde{B}_t \tag{9.21}$$

and so X is a solution to (9.17) on the probability space $(\Omega, \mathcal{F}, \widetilde{P})$. □

The upshot of Theorem 9.4 is that for, say, $t, x \mapsto a(t, x)$ bounded, a weak solution is produced under no regularity assumptions on $t, x \mapsto a(t, x)$ beyond measurability that is in fact necessary to make the equation meaningful. (We solved (9.17) earlier in a strong sense by converting it to an ODE, but that requires Lipschitz continuity in x.) That being said, not conclusion is made concerning uniqueness which may be harder to control without any structure of the coefficients.

The caution we exercised in constructing $\widetilde{P}$ an applying the conclusion of Girsanov's Theorem in the probability space $(\Omega, \mathcal{F}, \widetilde{P})$ is not without reason. Indeed, $\widetilde{P}$ is obtained by an extension argument and that may, and often will, wreak havoc on null sets. In fact, the measure $\widetilde{P}$ will typically not be absolutely continuous with respect to P due to the fact that B under $\widetilde{P}$ does not at all look like Brownian motion. Still, this is not because of its local behavior as $\widetilde{P}_T$ is still absolutely continuous with respect to the restriction of P to $\mathcal{F}_T^B$ for each $T > 0$. The source of the singularity is behavior at infinity; indeed, B may be transient to $+\infty$ under $\widetilde{P}$ while it is recurrent under P.

9.3 Brownian motion conditioned to stay positive.

We will now demonstrate the above issue on an example of Brownian motion conditioned to stay positive. Indeed, writing $\tau_0 := \inf\{t \geqslant 0\colon B_t = 0\}$ for Brownian motion with law P^x such that $P^x(B_0 = x) = 1$, we would like to describe the measure $P^x(\cdot|\tau_0 = \infty)$ for $x > 0$. Unfortunately, this is meaningless because $P^x(\tau_0 = \infty) = 0$ for all x. We will therefor proceed by a limit argument.

Continuing working in the Wiener space, given $T > 0$ and $x > 0$, consider the measure Q_t^x defined by

$$Q_T^x(A) := P(A|\tau_0 > T) \tag{9.22}$$

Since $P^x(\tau_0 > T) > 0$ for $x > 0$, this is well defined. We then claim:

Theorem 9.5 (Brownian motion conditioned to stay positive) *Let $(\Omega, \mathcal{F}, P)$ be the Wiener space. Then for all $x > 0$, $t \geqslant 0$ and $A \in \mathcal{F}_t$, the limit*

$$Q^x(A) := \lim_{T\to\infty} Q^x_T(A) \tag{9.23}$$

exists and extends to a unique probability measure on $(\Omega, \mathcal{F})$. Moreover, $\{B_t\colon t \geqslant 0\}$ under Q^x has the law of the 3-dimensional Bessel process started from x.

Proof. Write P^x for the law of the standard Brownian motion started from x and let E^x be the associated expectation. Define

$$h_t(x) := P^x(\tau_0 > t) \tag{9.24}$$

Then for each $0 \leqslant t < T$ and $A \in \mathcal{F}_t$, the Markov property gives

$$Q^x_T(A) = E^x\Bigl(1_{A\cap\{\tau_0>t\}}\frac{h_{T-t}(B_t)}{h_T(x)}\Bigr) \tag{9.25}$$

We now claim:

Lemma 9.6 (Reflection principle) *For each $t \geqslant 0$ and $x > 0$,*

$$h_t(x) = P^x(B_t > 0) - P^x(B_t < 0) \tag{9.26}$$

In particular, we have

$$\forall x \geqslant 0\, \forall t > 0\colon \quad \mathrm{e}^{-\frac{x^2}{2t}}\sqrt{\frac{2}{\pi}}\,\frac{x}{\sqrt{t}} \leqslant h_t(x) \leqslant \sqrt{\frac{2}{\pi}}\,\frac{x}{\sqrt{t}} + \frac{x^2}{t} \tag{9.27}$$

and

$$\forall x \geqslant 0\colon \quad \lim_{t\to\infty} \sqrt{t}\, h_t(x) = \sqrt{\frac{2}{\pi}}\, x \tag{9.28}$$

Postponing the proof of this lemma until the main line of argument is finished, plugging this into the ratio on the right of (9.25) we now observe that, for $T \geqslant 2t$, $T \geqslant 2$ and and $x \leqslant \sqrt{T}$,

$$\frac{h_{T-t}(B_t)}{h_T(x)} \leqslant \frac{\sqrt{\mathrm{e}}}{x}\frac{\sqrt{T}}{\sqrt{T-t}}\Bigl(B_t + 2\frac{B_t^2}{\sqrt{T-t}}\Bigr) \leqslant \frac{\sqrt{\mathrm{e}}}{x}(B_t + 2B_t^2) \tag{9.29}$$

where we relied on (9.27) along with $\sqrt{\pi/2} \leqslant 2$ and $\mathrm{e}^{-\frac{x^2}{2T}} \geqslant \mathrm{e}^{-1/2}$ when $x \leqslant \sqrt{T}$. The limit statement (9.28) in turn gives

$$\lim_{T\to\infty}\frac{h_{T-t}(B_t)}{h_T(x)} = \frac{B_t}{x} \tag{9.30}$$

The Dominated Convergence Theorem then allows us to conclude that

$$\lim_{T\to\infty} Q^x_T(A) = E^x\Bigl(1_{A\cap\{\tau_0>t\}}\frac{B_t}{B_0}\Bigr) \tag{9.31}$$

In particular, the limit in (9.23) exists. The argument used in the proof of Theorem 9.4 shows that Q^x extends to a probability measure on $(\Omega, \mathcal{F})$.

In order to identify the law of B under Q^x, we need a minor truncation argument. Pick $a \in (0, x)$, set $\tau_a := \inf\{t \geqslant 0 \colon B_t = a\}$ and use the Itô formula to write

$$\frac{B_t}{B_0} = \exp\bigl\{\log B_t - \log B_0\bigr\} = M_t \quad \text{on } \{\tau_a > t\} \tag{9.32}$$

where

$$M_t := \exp\biggl\{\int_0^t \frac{1}{B_s} 1_{\{\tau_a > s\}} \mathrm{d}B_s - \frac{1}{2}\int_0^t \frac{1}{B_s^2} 1_{\{\tau_a > s\}} \mathrm{d}s\biggr\} \tag{9.33}$$

Let $\widetilde{P}^x$ be the extension to $(\Omega, \mathcal{F})$ of $A \mapsto E^x(1_A M_t)$ on $\mathcal{F}_t^B$. Then

$$\forall A \in \mathcal{F}_t^B\colon \quad Q^x\bigl(A \cap \{\tau_a > t\}\bigr) = \widetilde{P}^x\bigl(A \cap \{\tau_a > t\}\bigr) \tag{9.34}$$

Since $s \mapsto B_s^{-1} 1_{\{\tau_a > s\}}$ is bounded, M is a martingale and Girsanov's Theorem shows that

$$\widetilde{B}_t := B_t - \int_0^t \frac{1}{B_s} 1_{\{\tau_a > s\}} \mathrm{d}s \tag{9.35}$$

is a standard Brownian motion under $\widetilde{P}^x$. Turning this around, this means that B satisfies

$$\mathrm{d}B_t = \frac{1}{B_t} 1_{\{\tau_a > t\}} \mathrm{d}t + \mathrm{d}\widetilde{B}_t \tag{9.36}$$

on $(\Omega, \mathcal{F}, \widetilde{P}^x)$. By (9.34), under Q^x the process $\{B_{\tau_a \wedge t} \colon t \geqslant 0\}$ is a 3-dimensional Bessel process stopped upon hitting level a. Since the 3-dimensional Bessel process does not hit zero in finite time, we have $\tau_a \to \infty$ as $a \downarrow 0$ under Q^x and so we get that $\{B_t \colon t \geqslant 0\}$ is a 3-dimensional Bessel process under Q^x. $\square$

Notice that Q^x is concentrated on paths that avoid zero yet P^x is not. This is actually quite common when exponential change of measure is invoked. A condition that ensures that the extension of measures $\widetilde{P}(A) := E(1_A M_t)$ remains absolutely continuous with respect to P is that M admits a terminal element; i.e., $M_t \to M_\infty$ in $L^1(P)$. Of course, this is not what usually happens in applications.

We also note that the expressions (9.25) and (9.31) where ratios of two random variables at different times appear are reminiscent of the h-transform from countable-state Markov chain theory. The role of these is exactly as there: express the change of weight of the random path due to the conditioning event.

9.4 Proof of Reflection Principle.

We owe to the reader:

Proof of Lemma 9.6. The identity (9.26) is classical, proved originally for random walks in consideration of so called Ballot Theorem by J.L.F. Bertrand in 1887 (with an earlier proof from 1878 due to W.A. Whitworth). We follow a continuum version of the elegant proof of the latter by Désiré André (also from 1887). The argument relies on a reflection trick; hence the title of the lemma.

The starting point is the disjoint decomposition

$$\{\tau_0 \leqslant t\} = \Bigl(\{\tau_0 \leqslant t\} \cap \{B_t \geqslant 0\}\Bigr) \cup \Bigl(\{\tau_0 \leqslant t\} \cap \{B_t < 0\}\Bigr) \tag{9.37}$$

Next observe that, by the strong Markov property, for $x > 0$,

$$P^x\big(\tau_0 \leqslant t \;\wedge\; B_t > 0\big) = E^x\big(1_{\{\tau_0\leqslant t\}} P^0(B_{t-s} > 0)|_{s:=\tau_0}\big) \tag{9.38}$$

Brownian symmetries imply that $P^0(B_{t-s} > 0) = P^0(B_{t-s} < 0)$ and so, wrapping this around, we conclude

$$P^x\big(\tau_0 \leqslant t \;\wedge\; B_t > 0\big) = P^x\big(\tau_0 \leqslant t \;\wedge\; B_t < 0\big) = P^x(B_t < 0) \tag{9.39}$$

where we also noticed that $B_t < 0$ implies $\tau_0 \leqslant t$ when $B_0 > 0$. Since $P^x(B_t = 0) = 0$, we thus get

$$\begin{aligned} P^x(\tau_0 > t) &= 1 - P^x(\tau_0 \leqslant t) \\ &= 1 - 2P^x(B_t < 0) = P^x(B_t > 0) - P^x(B_t < 0) \end{aligned} \tag{9.40}$$

thus proving (9.26).

In order to prove the remaining statements, we now write

$$\begin{aligned} P^x(B_t > 0) - P^x(B_t < 0) &= \int_0^\infty \frac{1}{\sqrt{2\pi t}}\big[\mathrm{e}^{-\frac{(y-x)^2}{2t}} - \mathrm{e}^{-\frac{(y+x)^2}{2t}}\big]\mathrm{d}y \\ &= \mathrm{e}^{-\frac{x^2}{2t}} \int_0^\infty \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{y^2}{2t}}\big[\mathrm{e}^{\frac{xy}{t}} - \mathrm{e}^{-\frac{xy}{t}}\big]\mathrm{d}y \end{aligned} \tag{9.41}$$

Writing the difference of the two exponentials as an integral yields

$$\begin{aligned} h_t(x) &= \sqrt{\frac{2}{\pi}}\,\frac{x}{t^{3/2}}\,\mathrm{e}^{-\frac{x^2}{2t}} \int_{-1}^1 \frac{1}{2}\Big(\int_0^\infty y\mathrm{e}^{-\frac{y^2}{2t}-\frac{yx}{t}s}\mathrm{d}y\Big)\mathrm{d}s \\ &= \sqrt{\frac{2}{\pi}}\,\frac{x}{\sqrt{t}}\,\mathrm{e}^{-\frac{x^2}{2t}} \int_0^1 \Big(\int_0^\infty z\mathrm{e}^{-\frac{z^2}{2}} \cosh\big(\tfrac{zx}{\sqrt{t}}s\big)\mathrm{d}z\Big)\mathrm{d}s \end{aligned} \tag{9.42}$$

where we changed variables via $y = z\sqrt{t}$ and then use the symmetry of the integral with respect to s to wrap the result using hyperbolic cosine. For the lower bound in (9.26) we now use that $\cosh(r) \geqslant 1$ while for the upper bound we bound $\cosh(r) \leqslant \mathrm{e}^r$ and then note that

$$\begin{aligned} \mathrm{e}^{-\frac{x^2}{2t}} \int_0^\infty z\mathrm{e}^{-\frac{z^2}{2}+\frac{zx}{\sqrt{t}}}\,\mathrm{d}z &\leqslant \int_0^\infty z\mathrm{e}^{-\frac{(z-x/\sqrt{t})^2}{2}}\mathrm{d}z \\ &\leqslant \int_{-x/\sqrt{t}}^\infty (u + \tfrac{x}{\sqrt{t}})\mathrm{e}^{-\frac{u^2}{2}}\,\mathrm{d}u \leqslant \mathrm{e}^{-\frac{x^2}{2t}} + \sqrt{\frac{\pi}{2}}\,\frac{x}{\sqrt{t}} \end{aligned} \tag{9.43}$$

Bounding the exponential by one, we then get the upper bound in (9.26). The limit statement (9.28) follows from (9.42) and the Dominated Convergence Theorem. □

Further reading: Karatzas-Shreve, Sections 2.6 and 3.5D