7. BROWNIAN MARTINGALES

The results in the previous chapter showed a particular role played by standard Brownian motion among all continuous local martingales. Here we examine this role in a somewhat different context.

7.1 Representing L^2-random variables.

We start by some observations. Recall that the Doob-Dynkin lemma states that if X and Y are random variables and X is $\sigma(Y)$ measurable, then $X = f(Y)$ for some Borel function f (between the respective spaces that we deliberately keep implicit). It follows that, if B is a standard Brownian motion and $\mathcal{F}_t^B := \sigma(B_s\colon s \leqslant t)$ is its natural filtration, then any $\mathcal{F}_t^B$-measurable random variable X is a measurable function of $\{B_s\colon s \leqslant t\}$. Leaving the technicalities aside, this should also somehow mean that X is a function of the infinitesimal increments $\{\mathrm{d}B_s\colon s < t\}$. This suggests a representation in terms of a stochastic integral, which we state in the form:

Theorem 7.1 *Let B be a Brownian motion endowed with natural filtration $\{\mathcal{F}_t^B\}_{t\geqslant 0}$. Then for each $X \in L^2$ there exists a stochastic process $Y \in \mathcal{V}_B$ such that*

$$\forall t \geqslant 0\colon\quad E(X|\mathcal{F}_t^B) = EX + \int_0^t Y_s \mathrm{d}B_s \quad \text{a.s.} \tag{7.1}$$

The process Y is determined uniquely up to the equivalence relation between processes in $\mathcal{V}_B$.

Proof. Let $X \in L^2$ and, shifting X by a constant, assume $EX = 0$. We start by the construction of Y which we first motivate as follows. Observe that $E(X|\mathcal{F}_t^B)$ is an element of the Hilbert space $L^2(\Omega, \mathcal{F}_t^B, P)$. The collection $\mathcal{I} := \{\int_0^t Y_s \mathrm{d}B_s\colon Y \in \mathcal{V}_B\}$, with a.s.-equal random variables identified as one, forms a closed linear subspace of $L^2(\Omega, \mathcal{F}_t^B, P)$. This suggests that we seek Y for which $\int_0^t Y_s \mathrm{d}B_s$ is nearest to $E(X|\mathcal{F}_t^B)$ in $\mathcal{I}$ or, alternatively, the orthogonal projection of $E(X|\mathcal{F}_t^B)$ onto $\mathcal{I}$.

Formulating the projection argument in terms of Hilbert spaces requires working with equivalence classes of random variables, so we rather proceed by mimicing the steps from Hilbert space theory for actual stochastic processes. Consider the quadratic functional

$$\varphi(Y) := E\left(\left[E(X|\mathcal{F}_t^B) - \int_0^t Y_s \mathrm{d}B_s\right]^2\right) \tag{7.2}$$

The above suggest looking for Y achieving $c := \inf\{\varphi(Y)\colon Y \in \mathcal{V}_B\}$. Pick a minimizing sequence $\{Y^{(n)}\}_{n\in\mathbb{N}} \in \mathcal{V}_B^{\mathbb{N}}$ (i.e., one for which $\varphi(Y^{(n)}) \to c$) and observe that the quadratic nature of φ implies

$$E\left(\left[\int_0^t Y_s^{(n)} \mathrm{d}B_s - \int_0^t Y_s^{(m)} \mathrm{d}B_s\right]^2\right) = 2\varphi(Y^{(n)}) + 2\varphi(Y^{(m)}) - 4\varphi\left(\frac{Y^{(n)} + Y^{(m)}}{2}\right) \tag{7.3}$$

The right hand side is not larger than $2\varphi(Y^{(n)}) + 2\varphi(Y^{(m)}) - 4c$ and so it tends to zero as $m, n \to \infty$. The left-hand side then tends to zero in this limit as well. The Itô isometry turns that conclusion into

$$\lim_{m,n\to\infty} E\left(\int_0^t (Y_s^{(n)} - Y_s^{(m)})^2 \mathrm{d}s\right) = 0. \tag{7.4}$$

It follows that $\{Y^{(n)}\}_{n\in\mathbb{N}}$ is Cauchy in $L^2([0,t]\times\Omega)$. As this holds for all $t \geqslant 0$, the Cauchy property holds even in $\mathcal{V}_B$.

Since $\mathcal{V}_B$ is the set of limits of all Cauchy sequences as above, it follows that there exists $Y \in \mathcal{V}_B$ such that $[\![Y^{(n)} - Y]\!]_B \to 0$. This implies $\int_0^t Y_s^{(n)}\mathrm{d}B_s \to \int_0^t Y_s \mathrm{d}B_s$ in L^2 and so $\varphi(Y^{(n)}) \to \varphi(Y)$. It follows that $\varphi(Y) = c$ and so Y is indeed a minimizer. Replacing Y by $Y + \epsilon Z$ for ϵ both positive and negative then shows

$$\forall t \geqslant 0\, \forall Z \in \mathcal{V}_B\colon\ E\left(\left[E(X|\mathcal{F}_t^B) - \int_0^t Y_s \mathrm{d}B_s\right]\int_0^t Z_s \mathrm{d}B_s\right) = 0. \tag{7.5}$$

This is actually we will need from Y in the sequel.

Fix $t \geqslant 0$ and let $0 = t_0 < \cdots < t_n = t$ be a partition of $[0,t]$. Define the complex-valued process

$$M_s := \exp\left\{\mathrm{i}\sum_{j=1}^n \lambda_j (B_{t_j\wedge s} - B_{t_{j-1}\wedge s}) + \frac{1}{2}\sum_{j=1}^n \lambda_j^2 (t_j \wedge s - t_{j-1}\wedge s)\right\} \tag{7.6}$$

and set

$$Z_s = \left(\mathrm{i}\sum_{j=1}^n \lambda_j 1_{(t_{j-1},t_j]}(s)\right) M_s \tag{7.7}$$

The Itô formula shows that $\mathrm{d}M_s = Z_s \mathrm{d}B_s$ and so

$$\int_0^t Z_s \mathrm{d}B_s = M_t - M_0 = M_t - 1. \tag{7.8}$$

Abbreviating

$$V_t := E(X|\mathcal{F}_t^B) - \int_0^t Y_s \mathrm{d}B_s \tag{7.9}$$

we thus have $E(V_t M_t) = E(V_t) = E(X) = 0$. This readily translates into

$$E\left(V_t \exp\left\{\mathrm{i}\sum_{j=1}^n \lambda_j' B_{t_j}\right\}\right) = 0 \tag{7.10}$$

where $\lambda_j' := \lambda_j - \lambda_{j-1}$. Since this holds for all $\lambda_1', \dots, \lambda_n'$, integrating this over all λ_j's against the Fourier transform of a function $f \in L^2(\mathbb{R}^n)$ shows

$$E\big(V_t f(B_{t_1}, \dots, B_{t_n})\big) = 0. \tag{7.11}$$

Specializing to indicators of bounded Borel sets is sufficient to conclude

$$E\big(V_t \,\big|\, \sigma(B_{t_1}, \dots, B_{t_n})\big) = 0 \quad \text{a.s.} \tag{7.12}$$

As the ordering of the t_i's no longer matters, this is true for all natural $n \geqslant 1$ and all $t_1, \dots, t_n \in [0,t]$.

Let $D_n := \{k2^{-n}t\colon k = 1, \dots, 2^n\}$ and set $D := \bigcup_{n\geqslant 0} D_n$. Note that $D_n \subseteq D_{n+1}$ gives $\sigma(B_u\colon u \in D_n) \subseteq \sigma(B_u\colon u \in D_{n+1})$ and $\sigma(\bigcup_{n\geqslant 1}\sigma(B_u\colon u \in D_n)) = \sigma(B_u\colon u \in D)$. We also readily check that $\sigma(B_u\colon u \in D) = \mathcal{F}_t^B$ by continuity of B. The Lévy Forward Theorem along with (7.12) then shows

$$E(V_t|\mathcal{F}_t^B) = 0 \quad \text{a.s.} \tag{7.13}$$

which by the fact that V_t is $\mathcal{F}_t^B$-measurable gives $V_t = 0$ a.s. Taking into account that we assumed $E(X) = 0$, this is the desired claim. □

Note that we can of course replace $E(X|\mathcal{F}_t^B)$ by X provided we assume that X is $\mathcal{F}_t^B$-measurable. The requirement $X \in L^2$ was quite useful in the proof but the Doob-Dynkin lemma seems to work without any assumption on integrability. A representation under just a.s. finiteness was established by Dudley in 1977; see Theorem 4.20 in Karatzas and Shreve. Note, however, that without some integrability, the uniqueness of the representation is lost. This is because, for each $t > 0$, one can find $Y \in \mathcal{V}_B^{\mathrm{loc}}$ such that

$$\int_0^t Y_s^2 \mathrm{d}s > 0 \quad \text{yet} \quad \int_0^t Y_s \mathrm{d}B_s = 0 \quad \text{a.s.} \tag{7.14}$$

We leave a proof of this fact to a homework exercise.

7.2 Extension to L^2-martingales.

The above theorem now rewrites for L^2-martingales as follows:

Theorem 7.2 *Let $(\Omega, \mathcal{F}, P)$ be a probability space supporting a standard Brownian motion B and a stochastic process M that is an L^2-martingale with respect to a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying $\forall t \geqslant 0\colon \mathcal{F}_t \subseteq \sigma(\mathcal{F}_t^B \cup \mathcal{N})$ for $\mathcal{F}_t^B := \sigma(B_s\colon s \leqslant t)$ and $\mathcal{N}$ the collection of all P-null sets. Then there exists $Y \in \mathcal{V}_B$, adapted to $\{\mathcal{F}_t^B\}_{t\geqslant 0}$, such that*

$$\forall t \geqslant 0\colon\; M_t = M_0 + \int_0^t Y_s \mathrm{d}B_s \quad \text{a.s.} \tag{7.15}$$

The process Y is determined uniquely up to the equivalence relation between processes in $\mathcal{V}_B$.

Proof. Given any natural $n \geqslant 1$, note that

$$E(M_n|\mathcal{F}_t^B) = E(M_n|\mathcal{F}_t) = M_{t\wedge n} \quad \text{a.s.} \tag{7.16}$$

by the martingale property and the fact that each $A \in \mathcal{F}_t^B$ differs from some $A' \in \mathcal{F}_t^B$ by a null set. Similarly,

$$M_0 = E(M_0) \quad \text{a.s.} \tag{7.17}$$

by the fact that P is trivial on $\mathcal{F}_0$. With this in mind, Theorem 7.1 yields existence of $Y^{(n)} \in \mathcal{V}_B$, adapted to $\{\mathcal{F}_t^B\}_{t\geqslant 0}$, such that

$$\forall t \geqslant 0\colon \quad M_{t\wedge n} = M_0 + \int_0^t Y_s^{(n)} \mathrm{d}B_s \quad \text{a.s.} \tag{7.18}$$

Taking this for n replaced by $n+1$ at $t = n$, we get

$$\int_0^n Y_s^{(n+1)} \mathrm{d}B_s = \int_0^n Y_s^{(n)} \mathrm{d}B_s \quad \text{a.s.} \tag{7.19}$$

In light of square integrability, the Itô isometry then gives

$$\int_0^n \big(Y_s^{(n+1)} - Y_s^{(n)}\big)^2 \mathrm{d}s = 0 \tag{7.20}$$

meaning that the two processes are equivalent as elements of $L^2([0,n]\times\Omega)$. Setting

$$Y_t := \sum_{n\geqslant 1} Y_t^{(n)} 1_{(n-1,n]}(t) \tag{7.21}$$

this means that we can replace $Y^{(n)}$ by $\{Y_s 1_{[0,n]}(s)\colon s\geqslant 0\}$ in (7.18) without affecting the a.s. equality. The indicator is removed by taking $n\to\infty$ with the help of Itô isometry and Dominated convergence, which then yields the desired claim. □

Note that we have note made any assumption of continuity of the martingale which is mainly because the proof is done for each time separately. Notwithstanding, the result gives continuity of M as a corollary:

Corollary 7.3 *Let M be an L^2-martingale with respect to the filtration $\{\widetilde{\mathcal{F}}_t^B\}_{t\geqslant 0}$, where B is a standard Brownian motion and $\widetilde{\mathcal{F}}_t^B := \sigma(\mathcal{F}_t^B \cup \mathcal{N})$, for $\mathcal{F}_t^B := \sigma(B_s\colon s\leqslant t)$ and $\mathcal{N}$ the collection of all P-null sets. Then M admits a continuous version.*

Proof. By Theorem 7.2, $\{M_0 + \int_0^t Y_s \mathrm{d}B_s\colon t\geqslant 0\}$ is a version of M. Since, for our choice of the filtration, the stochastic integrals admit a continuous version, so does M. □

Recall that, by Lemma 1.1 that a.e. sample of a martingale has left and right limits along rationals at all (positive) times. For filtrations that are one-sided continuous, the corresponding one-sided limit is a version of the martingale. However, this does not imply that for filtrations that are continuous the martingale is continuous. (Indeed, consider a Poisson process with its augmented filtration.) So Corollary 7.3 does say something more than general arguments seem to imply.

Further reading: Karatzas-Shreve, Section 3.4D