6. Representation theorems

Here we present our first applications of stochastic calculus and Itô formula. All of these related to discovering a standard Brownian motion in the structure of a continuous local martingale. It is here where we find it beneficial that the stochastic integral has been extended to continuous local martingales.

6.1 Lévy characterization.

We start with the following cute observation going back to P. Lévy:

Theorem 6.1 (Lévy's characterization of Brownian motion) *Let $M \in \mathscr{M}_{\text{loc}}^{\text{cont}}$ be such that $\forall t \geqslant 0\colon \langle M\rangle_t = t$ and $M_0 = 0$. Then M is a standard Brownian motion.*

Proof. Given $M \in \mathscr{M}_{\text{loc}}^{\text{cont}}$ and $\lambda \in \mathbb{R}$, let $\{Z_t\colon t \geqslant 0\}$ be the $\mathbb{C}$-valued process defined by

$$Z_t := \mathrm{e}^{\mathrm{i}\lambda M_t + \frac12 \lambda^2 \langle M\rangle_t}. \tag{6.1}$$

The Itô formula then shows

$$\mathrm{d}Z_t = Z_t\Bigl(\mathrm{i}\lambda \mathrm{d}M_t + \frac12\lambda^2\mathrm{d}\langle M\rangle_t\Bigr) + \frac12\lambda^2 Z_t\mathrm{d}\langle M\rangle_t = \mathrm{i}\lambda Z_t \mathrm{d}M_t. \tag{6.2}$$

As the right-hand side has no "drift term," we conclude $Z \in \mathscr{M}_{\text{loc}}^{\text{cont}}$. Under the assumption that $\langle M\rangle_t$ the random variable $|Z_t|$ is bounded by $\mathrm{e}^{\frac12\lambda^2 t}$. As bounded local martingales are martingales, we get that $Z \in \mathscr{M}^{\text{cont}}$.

The fact that Z is a martingale means that $E(Z_t|\mathcal{F}_s) = Z_s$ for all $0 \leqslant s \leqslant t$. For the case at hand this reads

$$E\bigl(\mathrm{e}^{\mathrm{i}\lambda M_t + \frac12\lambda^2 t}\,\big|\,\mathcal{F}_s\bigr) = \mathrm{e}^{\mathrm{i}\lambda M_s + \frac12\lambda^2 t}. \tag{6.3}$$

Rearranging with the help of the $\mathcal{F}_s$-measurability then gives

$$E\bigl(\mathrm{e}^{\mathrm{i}\lambda(M_t - M_s)}\,\big|\,\mathcal{F}_s\bigr) = \mathrm{e}^{-\frac12\lambda^2(t-s)}. \tag{6.4}$$

Using this iteratively along the sequence $0 = t_0 < \dots < t_n$ shows that

$$E\biggl(\exp\Bigl\{\mathrm{i}\sum_{j=1}^n \lambda_j(M_{t_j} - M_{t_{j-1}})\Bigr\}\biggr) = \exp\Bigl\{-\frac12\sum_{j=1}^n \lambda_j(t_j - t_{j-1})\Bigr\} \tag{6.5}$$

holds for all $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. Using the Cramér-Wold device, it follows that M has the same finite-dimensional distributions as the standard Brownian motion. Since M is continuous with M_0, it is a standard Brownian motion. □

We note that the above has been proved in 275D for the case that M is a stochastic integral with respect to standard Brownian motion. The fact that we can now treat integrals with respect to general continuous local martingales allows us to prove this without any restriction on the structure of M; see also Theorem 6.2 below.

We also remark that the result extends seamlessly to $\mathbb{R}^d$-valued local martingales M. The condition we then need is that the Cartesian components $M^{(1)}, \dots, M^{(d)}$ of M obey

$$\forall t \geqslant 0\ \forall i,j = 1,\dots,d\colon\ \bigl\langle M^{(i)}, M^{(j)}\bigr\rangle_t = t\delta_{ij} \tag{6.6}$$

The argument is identical modulo introduction of a dot product in relevant places.

A natural question arises what happens when $\langle M\rangle_t = t$ is not assumed. Here is one type of a result that one can hope to get in such a case:

Theorem 6.2 (Time change to Brownian motion) *Let $M \in \mathscr{M}_{\text{loc}}^{\text{cont}}$ be such that $M_0 = 0$ and every path of $t \mapsto \langle M\rangle_t$ is strictly increasing with $\lim_{t\to\infty}\langle M\rangle_t = \infty$. For each $t \geqslant 0$, set*

$$T(t) := \inf\bigl\{u \geqslant 0\colon \langle M\rangle_u \geqslant t\bigr\}. \tag{6.7}$$

Then $T(t)$ is a stopping time for each $t \geqslant 0$, every path of $t \mapsto T(t)$ is continuous and the process $\{B_t\colon t \geqslant 0\}$, defined by $B_t := M_{T(t)}$, is a standard Brownian motion. Moreover, we have

$$\forall t \geqslant 0\colon\;\; M_t = B_{\langle M\rangle_t}. \tag{6.8}$$

Proof. That $T(t)$ is a stopping time follows from $\{T(t) \leqslant u\} = \{\langle M\rangle_u \leqslant t\}$ as implied by continuity of $\langle M\rangle$. The continuity of T is then inherited from the continuity and strict monotonicity of $\langle M\rangle$. In order to prove the main part of the claim, note that

$$\forall t \geqslant 0\colon\;\; \langle M\rangle_{T(t)} = t \tag{6.9}$$

which also entails $T(t) \to \infty$ as $t \to \infty$. Since the process $\{Z_t\colon t \geqslant 0\}$ from (6.1) continues to be a local martingale, the explicit form

$$Z_{T(t)\wedge u} = \mathrm{e}^{\mathrm{i}\lambda M_{T(t)\wedge u} + \frac12\langle M\rangle_{T(t)\wedge u}} \tag{6.10}$$

reveals that $\{Z_{T(t)\wedge u}\colon u \geqslant 0\}$ is a bounded martingale. For $s \leqslant t$, the Optional Stopping Theorem applied to the stopping times $T(s) \leqslant T(t)$ under the filtration $\{\mathcal{F}_{T(t)}\colon t \geqslant 0\}$ then gives

$$E\bigl(Z_{T(t)}\,\big|\,\mathcal{F}_{T(s)}\bigr) = Z_{T(s)} \tag{6.11}$$

Using (6.9), this now readily translates into

$$E\bigl(\mathrm{e}^{\mathrm{i}\lambda(M_{T(t)} - M_{T(s)})}\,\big|\,\mathcal{F}_{T(s)}\bigr) = \mathrm{e}^{-\frac12\lambda^2(t-s)}. \tag{6.12}$$

Proceeding as in the proof of Lévy characterization, we then readily conclude that the process $\{M_{T(t)}\colon t \geqslant 0\}$ is a standard Brownian motion. The identity (6.8) is a direct consequence of T being the inverse of $\langle M\rangle$. $\square$

Theorem 6.2 makes a number of convenient assumptions that can be further relaxed. First, we do not need to assume that the various assumed properties occur for all paths, but rather only almost surely. Here we need to impose the assumption that $\mathcal{F}_0$ contains all P-null sets and modify the definitions of B on a null set.

Another assumption that is easy to drop is $\langle M\rangle_t \to \infty$ as $t \to \infty$, which ensures that the intrinsic time of $\{M_{T(t)}\colon t \geqslant 0\}$ varies throughout all the positive reals. If this is not assumed, we only get a standard Brownian motion up to a stopping time; the standard trick is then to enhance the probability space and append another path of standard Brownian motion after that stopping time.

A somewhat more difficult assumption is that of strict monotonicity of $t \mapsto \langle M\rangle_t$ which ensures the uniqueness of the inversion. Since the Optional Stopping Theorem for continuous martingales allows us to work with right-continuous filtrations, here one modifies the definition of $T(t)$ to make it right-continuous (which is achieved by replacing "$\leqslant t$" in (6.7) by "$> t$"). The general time-change result of above type can be found as Theorem 4.6 in Karatzas and Shreve.

As a final remark concerning the above results, we welcome the reader to compare them with the so called *Skorokhod embedding* which says that every discrete time martingale can be embedded into a path of standard Brownian motion using a sequence of stopping times. Besides elegance, this fact is very useful in proving the so called Martingale Functional Central Limit Theorem.

6.2 Representation via a stochastic integral.

While time change to Brownian motion is definitely a very useful tool, at times it suffices to represent the continuous martingale only as a stochastic integral with respect to Brownian motion. Here is a result in this vain:

Theorem 6.3 *Given a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ with $\mathcal{F}_0$ containing all P-null sets, let $M \in \mathscr{M}_{\text{loc}}^{\text{cont}}$ be adapted to $\{\mathcal{F}_t\}_{t\geqslant 0}$ and such that $t \mapsto \langle M\rangle_t$ is absolutely continuous a.s. Unless $\langle M\rangle$ is strictly increasing a.s., suppose in addition that the probability space supports a standard Brownian motion $\{\widetilde{B}_t\colon t \geqslant 0\}$ which is adapted to $\{\mathcal{F}_t\}_{t\geqslant 0}$ and independent of M. Then there exists a standard Brownian motion $\{B_t\colon t \geqslant 0\}$ and a version of the Lebesgue derivative of $s \mapsto \langle M\rangle_s$ that lies in $\mathcal{V}_B^{\text{loc}}$ such that*

$$\forall t \geqslant 0\colon \quad M_t = M_0 + \int_0^t \sqrt{\frac{\mathrm{d}\langle M\rangle_s}{\mathrm{d}s}}\, \mathrm{d}B_s \quad \text{a.s.} \tag{6.13}$$

In short, every continuous local martingale with absolutely continuous quadratic variation is an Itô integral with respect to standard Brownian motion.

For the proof, we need:

Lemma 6.4 (Substitution rule for Itô integrals) *Given $M \in \mathscr{M}_{\text{loc}}^{\text{cont}}$ and $X \in \mathcal{V}_M^{\text{loc}}$, assuming the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ is such that $\mathcal{F}_0$ contains all P-null sets, let $N \in \mathscr{M}_{\text{loc}}^{\text{cont}}$ be defined as*

$$\forall t \geqslant 0\colon \; N_t = \int_0^t X_s \mathrm{d}M_s \quad \text{a.s.} \tag{6.14}$$

Then for all $Y \in \mathcal{V}_N^{\text{loc}}$ we have $XY \in \mathcal{V}_M^{\text{loc}}$ and

$$\forall t \geqslant 0\colon \; \int_0^t X_s Y_s \mathrm{d}M_s = \int_0^t Y_s \mathrm{d}N_s \quad \text{a.s.} \tag{6.15}$$

In short, the substitution rule $\mathrm{d}N_t = X_s \mathrm{d}M_s$ applies.

Proof. Suppose first that $Y_s := Y_u 1_{(u,v]}(s)$ with Y_u bounded. Then for $X \in \mathcal{V}_0$ such that (without loss of generality) u and v belong among the partition points, we readily check

$$\int_0^t X_s Y_s \mathrm{d}M_s = Y_u \int_u^v X_s \mathrm{d}M_s = Y_u (N_{t\wedge v} - N_{t\wedge v}) \tag{6.16}$$

using the explicit formula for the integral. (We suppress the details that we leave to the reader.) Next take a sequence $\{X^{(n)}\} \in \mathcal{V}_0^{\mathbb{N}}$ with $\int_0^t (X_s^{(n)} - X_s)^2 \mathrm{d}\langle M\rangle \to 0$ in probability for all $t \geqslant 0$. Lemma 5.5 then gives

$$\forall r \geqslant 0\colon \quad \int_0^r X_s^{(n)} \mathrm{d}M_s \;\underset{n\to\infty}{\overset{P}{\longrightarrow}}\; \int_0^r X_s \mathrm{d}M_s = N_r \tag{6.17}$$

and, since also $\int_0^t Y_s^2 (X_s^{(n)} - X_s)^2 \mathrm{d}\langle M\rangle \to 0$ in probability, we similarly get

$$\forall t \geqslant 0\colon \quad \int_0^t X_s^{(n)} Y_s \mathrm{d}M_s \;\underset{n\to\infty}{\overset{P}{\longrightarrow}}\; \int_0^t X_s Y_s \mathrm{d}M_s \tag{6.18}$$

It follows (6.16) holds a.s. for each $X \in \mathcal{V}_M^{\mathrm{loc}}$.

Combining (6.16) with

$$Y_u (N_{t\wedge v} - N_{t\wedge v}) = \int_0^t Y_s \mathrm{d}N_s \tag{6.19}$$

proves (6.15) for Y as above. Additivity then extends this to all $Y \in \mathcal{V}_0$. We now perform another extension by picking up $\{Y^{(n)}\} \in \mathcal{V}_0$ such that $\int_0^t (Y_s^{(n)} - Y_s)^2 \mathrm{d}\langle N\rangle_s \to 0$ in probability for each $t \geqslant 0$. (This is possible because $Y \in \mathcal{V}_N^{\mathrm{loc}}$.) Using the substitution rule

$$\mathrm{d}\langle N\rangle_s = X_s^2 \mathrm{d}\langle M\rangle_s \tag{6.20}$$

for ordinary Lebesgue-Stieltjes integrals, this gives $\int_0^t X_s^2 (Y_s^{(n)} - Y_s)^2 \mathrm{d}\langle M\rangle_s \to 0$ in probability for each $t \geqslant 0$ showing that $XY \in \mathcal{V}_M$. The equality $\int_0^t X_s Y_s^{(n)} \mathrm{d}M_s = \int_0^t Y_s^{(n)} \mathrm{d}N_s$ proved earlier, then yields (6.15) for all $Y \in \mathcal{V}_N^{\mathrm{loc}}$ as desired. $\square$

We are now ready to give:

Proof of Theorem 6.3. Let $(\Omega, \mathcal{F}, P)$ be a probability space supporting a continuous local martingale $\{M_t \colon t \geqslant 0\}$ and a standard Brownian motion $\{\widetilde{B}_t \colon t \geqslant 0\}$ with both of these adapted to a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ such that $\mathcal{F}_0$ contains all P-null sets. Let

$$\Omega_0 := \bigl\{t \mapsto \langle M\rangle_t \text{ is AC}\bigr\} \tag{6.21}$$

and note that, since absolute continuity of $\langle M\rangle$ amounts to a countable number of conditions involving differences of $\langle M\rangle$ over intervals of time with rational endpoints, we have $\Omega_0 \in \mathcal{F}$. By our assumptions, $\Omega_0 \in \mathcal{F}_0$.

On Ω_0, for each $t > 0$ abbreviate

$$\widetilde{Y}_t := \liminf_{n\to\infty} \bigl(\langle M\rangle_t - \langle M\rangle_{t-1/n}\bigr) n \tag{6.22}$$

and for all $t \geqslant 0$ set

$$Y_t := \begin{cases} \widetilde{Y}_t, & \text{if } t > 0 \,\wedge\, \widetilde{Y}_t < \infty, \\ 0, & \text{else.} \end{cases} \tag{6.23}$$

We put $Y_t := 0$ for all $t \geqslant 0$ on Ω_0. Clearly, Y is non-negative, jointly measurable and, thanks to the use of a left limit, adapted to $\{\mathcal{F}_t\}_{t\geqslant 0}$. Moreover, Y_t is the left derivative of $\langle M\rangle$ at t whenever this derivative exists. The absolute continuity along with the

Lebesgue differentiation theorem then give

$$\forall t \geqslant 0\colon \quad \int_0^t Y_s\,\mathrm{d}s = \langle M\rangle_t \quad \text{ on } \Omega_0. \tag{6.24}$$

In particular, Y is an adapted, jointly measurable version of $\frac{\mathrm{d}\langle M\rangle_t}{\mathrm{d}t}$.

Next define $\{B_t \colon t \geqslant 0\}$ by

$$B_t := \int_0^t \frac{1}{\sqrt{Y_s}} 1_{\{Y_s>0\}}\,\mathrm{d}M_s + \int_0^t 1_{\{Y_s=0\}}\mathrm{d}\widetilde{B}_s, \tag{6.25}$$

where we assumed the use of continuous versions of the stochastic integrals. The properties of the stochastic integral imply $B \in \mathscr{M}_{\text{loc}}^{\text{cont}}$ with

$$\langle B\rangle_t = \int_0^t \Bigl(\frac{1}{\sqrt{Y_s}}\Bigr)^2 1_{\{Y_s>0\}}\,\mathrm{d}\langle M\rangle_s + \int_0^t 1_{\{Y_s=0\}}\mathrm{d}s \tag{6.26}$$

On Ω_0 we have $Y_s = \frac{\mathrm{d}\langle M\rangle_s}{\mathrm{d}t}$ at Lebesgue a.e. $s \geqslant 0$, which allows to perform the substitution $\mathrm{d}\langle M\rangle_s = Y_s\mathrm{d}s$ showing that the first integral equals $\int_0^t 1_{\{Y_s>0\}}\mathrm{d}s$. Combining this with the second integral, we conclude $\langle B\rangle_t = t$ a.s. (The null set does not depend on t by continuity of both sides.) By Theorem 6.1, B is a standard Brownian motion.

To conclude the proof, we now observe that, by a mild extension of the substitution rule from Lemma 6.4,

$$\begin{aligned}\int_0^t \sqrt{Y_s}\,\mathrm{d}B_s &= \int_0^t \sqrt{Y_s}\frac{1}{\sqrt{Y_s}} 1_{\{Y_s>0\}}\,\mathrm{d}M_s + \int_0^t \sqrt{Y_s} 1_{\{Y_s=0\}}\mathrm{d}\widetilde{B}_s \\ &= \int_0^t \sqrt{Y_s}\frac{1}{\sqrt{Y_s}} 1_{\{Y_s>0\}}\,\mathrm{d}M_s = \int_0^t 1_{\{Y_s>0\}}\mathrm{d}M_s,\end{aligned} \tag{6.27}$$

where we dropped the second integral on the right of the first line because the integrand vanishes and then simplified the integrand in the first integral. In light of the fact that

$$\int_0^t (1-1_{\{Y_s>0\}})^2\mathrm{d}\langle M\rangle_s = \int_0^t 1_{\{Y_s=0\}}Y_s\mathrm{d}s = 0 \quad \text{on } \Omega_0 \tag{6.28}$$

the very last integral in (6.27) equals $\int_0^t \mathrm{d}M_s = M_t - M_0$ a.s. This proves (6.13) and thus the whole claim. □

Note that the role of the auxiliary Brownian motion $\widetilde{B}$ is to make the variance grow even on intervals where $\langle M\rangle$, and thus also M are constant. No such intervals exist when $\langle M\rangle$ is strictly increasing throughout, in which case $\widetilde{B}$ is not needed.

We also note that the above theorem extends to $\mathbb{R}^d$-valued martingales. An additional difficulty is that the quadratic variation $\langle M, M\rangle$ is matrix valued and so is thus its adapted, jointly-measurable time derivative Y. We then strive to write $\mathrm{d}M_t = U_t D_t \mathrm{d}B_t$ where D_t is a diagonal matrix and U_t is an orthogonal matrix such that $Y_t = U_t(D_t)^2 U_t^{+}$. This is done by polar decomposition with a pesky detail is that we need also D and U to be adapted and jointly measurable. We refer the reader to Theorem 4.2 in Section 3.4A of Karatzas-Shreve.

Further reading: Section 3.3B and 3.4A of Karatzas-Shreve