4. Integrability and localization

Here we will continue discussing the Itô integral with respect to continuous martingales. The topics we will focus on are criteria for integrability and extension of the integral to local martingales and locally integrable processes.

4.1 Which processes can we integrate?.

The stochastic integral would hardly be very useful if we cannot supply an independent characterization of the class of processes that can be integrated. The answer to this turns out to be somewhat more subtle than for the Itô integral with respect to standard Brownian motion. Indeed, it will matter whether the quadratic variation process is absolutely continuous a.s. or not. The former case is actually the same:

Proposition 4.1 *Let $M \in \mathscr{M}_2^{\text{cont}}$ and recall the definition of $\mathcal{V}_M$ from (3.12). If $t \mapsto \langle M\rangle_t$ is absolutely continuous a.s., then $\overline{\mathcal{V}_0}^{[\![\cdot]\!]_M} = \mathcal{V}_M$.*

Proof. Let $Y \in \mathcal{V}_M$ and suppose first that Y is bounded by some $K > 0$. A lemma from 275D gives existence of a sequence $\{Y^{(n)}\}_{n\in\mathbb{N}} \in \mathcal{V}_0$ such that

$$\forall t \geqslant 0\colon\ E\left(\int_0^t (Y_s - Y_s^{(n)})^2 \mathrm{d}s\right) \underset{n\to\infty}{\longrightarrow} 0. \tag{4.1}$$

We may also assume that the processes $\{Y^{(n)}\}_{n\in\mathbb{N}}$ are bounded by the same constant as Y. Resorting to a subsequence if necessary, a Borel-Cantelli argument in turn permits us to assume that

$$\Big\{t \geqslant 0\colon\ \limsup_{n\to\infty} |Y_t - Y_t^{(n)}| > 0\Big\} \tag{4.2}$$

has vanishing Lebesgue measure P-a.s.

The assumed absolute continuity of $t \mapsto \langle M\rangle_t(\omega)$ now implies the existence of a (random) Radon-Nikodym derivative $\frac{\mathrm{d}\langle M\rangle_t}{\mathrm{d}t}$ which is a locally Lebesgue integrable function $F_M\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\int_0^t (Y_s - Y_s^{(n)})^2 \mathrm{d}\langle M\rangle_s = \int_0^t (Y_s - Y_s^{(n)})^2 F_M(s)\mathrm{d}s \tag{4.3}$$

holds for each $t \geqslant 0$. In light of the Lebesgue-null property of the set (4.2) and the boundedness of $Y - Y^{(n)}$, the integral on the right converges to zero P-a.s. by the Dominated Convergence Theorem. But the integral on the left is bounded by $4K\langle M\rangle_t$, which is in L^1. The Dominated Convergence Theorem then shows

$$\forall t \geqslant 0\colon\ E\left(\int_0^t (Y_s - Y_s^{(n)})^2 \mathrm{d}\langle M\rangle_s\right) \underset{n\to\infty}{\longrightarrow} 0 \tag{4.4}$$

proving that $[\![Y - Y^{(n)}]\!]_M \to 0$ and thus $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}$ whenever $Y \in \mathcal{V}_M$ is bounded.

If $Y \in \mathcal{V}_M$ is unbounded, we set $Y_s^{(n)} = Y_s 1_{\{|Y_s|\leqslant n\}}$ and observe that $Y^{(n)} \in \mathcal{V}_M$ is bounded and so $Y^{(n)} \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}$ by the previous argument. But then

$$E\biggl(\int_0^t (Y_s - Y_s^{(n)})^2 \mathrm{d}\langle M\rangle_s\biggr) = E\biggl(\int_0^t Y_s^2 1_{\{|Y_s|>n\}} \mathrm{d}\langle M\rangle_s\biggr) \tag{4.5}$$

and the right-hand side tends to zero as $n \to \infty$ by the Monotone Convergence Theorem and the fact that $Y \in \mathcal{V}_M$. Hence $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}$ in this case as well. □

For the case when $t \mapsto \langle M\rangle_t$ is not absolutely continuous, we will have to rely on the following weaker conclusion:

Proposition 4.2 *For all $M \in \mathscr{M}_2^{\mathrm{cont}}$ and all $Y \in \mathcal{V}_M$ progressively measurable, $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}$.*

Proof. The proof proceeds by a random time change that, effectively, reduces it again to the case (4.1) above. Progressive measurability is needed to ensure that the time-changed process is adapted to the time-changed filtration.

A natural process to base the time change on is $t \mapsto \langle M\rangle_t$. Unfortunately, this process may not be a strictly increasing (and thus one-to-one) function and so we instead work with $t \mapsto \langle M\rangle_t + t$. The inverse is supplied by the stopping times

$$T(u) := \inf\bigl\{t \geqslant 0 \colon \langle M\rangle_t + t \geqslant u\bigr\} \tag{4.6}$$

for which we get

$$\forall u \geqslant 0\colon\ T(u) + \langle M\rangle_{T(u)} = u. \tag{4.7}$$

The process $t \mapsto T(u)$ is continuous and strictly increasing.

Let now $Y \in \mathcal{V}_M$ be as in the statement. The argument in the previous proof permits us to assume that Y is bounded, say, $|Y|_t \leqslant K$ for all $t \geqslant 0$. The progressive measurability ensures that $\{Y_{T(u)} \colon u \geqslant 0\}$ is adapted to the filtration $\{\mathcal{F}_{T(u)}\}_{u\geqslant 0}$ and the joint measurability of $\{Y_t \colon t \geqslant 0\}$ and $\{T(u) \colon u \geqslant\}$ (implied, in the latter case, by continuity) shows that $\{Y_{T(u)} \colon u \geqslant 0\}$ is jointly measurable. Since $E[\int_0^t Y_s^2 \mathrm{d}s] \leqslant Kt$ for each $t \geqslant 0$, the aforementioned lemma from 275D yields existence of processes $\{Y^{(n)}\}_{n\in\mathbb{N}}$ of the form

$$Y_u^{(n)} = Z_0^{(n)} 1_{\{0\}}(u) + \sum_{i=1}^{m_n} Z_i^{(n)} 1_{(t_{i-1}^n, t_i^n]}(u) \tag{4.8}$$

with $m \in \mathbb{N}$, $0 = t_0^n < \dots < t_{m_n}$ such that

$$\forall n \in \mathbb{N}\ \forall i = 0,\dots,m_n\colon\ Z_i^{(n)} \text{ is } \mathcal{F}_{T(t_{i-1}^n \vee 0)}\text{-measurable} \tag{4.9}$$

for which

$$\forall t \geqslant 0\colon\ E\biggl(\int_0^t (Y_{T(u)} - Y_u^{(n)})^2 \mathrm{d}u\biggr) \underset{n\to\infty}{\longrightarrow} 0. \tag{4.10}$$

Abbreviating

$$\widetilde{Y}_s^{(n)} := Y_{\langle M\rangle_s + s}^{(n)} \tag{4.11}$$

which is jointly measurable thanks to the fact that $s \mapsto \langle M\rangle_s$ is jointly measurable, the substitution $u = \langle M\rangle_s + s$ inside the integral then shows, for each $t \geqslant 0$, that

$$
\begin{aligned}
E\biggl(\int_0^t (Y_s-\widetilde{Y}_s^{(n)})^2 \mathrm{d}\langle M\rangle_s\biggr) &\leqslant E\biggl(\int_0^t (Y_s - Y^{(n)}_{\langle M\rangle_s+s})^2(\mathrm{d}\langle M\rangle_s+\mathrm{d}s)\biggr)\\
&= E\biggl(\int_0^{t+\langle M\rangle_t} (Y_{T(u)}-Y_u^{(n)})^2\mathrm{d}u\biggr) \qquad (4.12)\\
&\leqslant E\biggl(\int_0^{t+v} (Y_{T(u)}-Y_u^{(n)})^2\mathrm{d}u\biggr) + 4K^2E\bigl((t+\langle M_t\rangle)1_{\{\langle M\rangle_t\geqslant v\}}\bigr),
\end{aligned}
$$

where we used that $(Y_{T(u)}-Y_u^{(n)})^2 \leqslant 4K^2$. The first term on the right now tends to zero as $n\to\infty$ by (4.10) while the second term tends to zero as $v\to\infty$ using the Dominated Convergence Theorem. We conclude that $[\![Y-\widetilde{Y}^{(n)}]\!]_M \to 0$.

To get the claim it thus suffices to show that $\widetilde{Y}^{(n)} \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}$ for each $n \geqslant 0$. For this we observe that the continuity and strict monotonicity of $t \mapsto T(t)$ gives

$$
\widetilde{Y}_u^{(n)} = Z_0^{(n)}1_{\{0\}}(u) + \sum_{i=1}^{m_n} Z_i^{(n)} 1_{(T(t_{i-1}^n),T(t_i^n)]}(u). \qquad (4.13)
$$

Defining $T_k(t) := 2^{-k}\lceil 2^kT(t)\rceil$, we have $T_k(t) \downarrow T(t)$ and so $1_{(T_k(s),T_k(u)]} \to 1_{(T(s),T(u)]}$ pointwise as $k\to\infty$ for all $s<u$. It follows that, as $k\to\infty$,

$$
\widetilde{Y}_u^{(n,k)} := Z_0^{(n)}1_{\{0\}}(u) + \sum_{i=1}^{m_n} Z_i^{(n)} 1_{(T_k(t_{i-1}^n),T_k(t_i^n)]}(u) \qquad (4.14)
$$

converges to $\widetilde{Y}_u^{(n)}$ for each $u \geqslant 0$. The limit $[\![\widetilde{Y}^{(n,k)}-\widetilde{Y}^{(n)}]\!]_M \to 0$ as $k\to\infty$ then takes place by the Dominated Convergence Theorem.

We will now show that $Y^{(n,k)} \in \mathcal{V}_0$. Indeed, thanks to the dyadic approximation of the stopping times, $(T_k(t_{i-1}^n), T_k(t_i^n)]$ is the union of intervals $(2^{-k}(j-1), 2^{-k}j]$ for j such that $T_k(t_{i-1}^n) < 2^{-k}j \leqslant T_k(t_i^n)$. Noting that $T_k(t_i^n) \leqslant t_{m_n}^n$ and setting $r_{n,k} := \lceil 2^k t_{m_n}^n\rceil$, we can thus rewrite the right-hand side of (4.14) as

$$
\widetilde{Y}_u^{(n,k)} = Z_0^{(n)}1_{\{0\}}(u) + \sum_{i=1}^{m_n}\sum_{j=1}^{r_{n,k}} \bigl(Z_i^{(n)} 1_{\{T_k(t_{i-1}^n)<2^{-k}j\leqslant T_k(t_i^n)\}}\bigr) 1_{(2^{-k}(j-1),2^{-k}j]}(u). \qquad (4.15)
$$

Now observe that $\{T_k(t_{i-1}^n) < 2^{-k}j \leqslant T_k(t_i^n)\} \in \mathcal{F}_{2^{-k}(j-1)}$ and that $Z_i^{(n)}$ is $\mathcal{F}_{t_{i-1}^n}$-measurable and thus also $\mathcal{F}_{2^k\lceil t_{i-1}^n\rceil}$-measurable. Hence we get that

$$
Z_i^{(n)} 1_{\{T_k(t_{i-1}^n)<2^{-k}j\leqslant T_k(t_i^n)\}} \text{ is } \mathcal{F}_{2^{-k}(j-1)}\text{-measurable} \qquad (4.16)
$$

for each $i = 1,\dots,m_n$ and each $j = 1,\dots,r_{n,k}$. This implies $\widetilde{Y}^{(n,k)} \in \mathcal{V}_0$ for each $n,k \geqslant 0$ and, by above reasoning, $\widetilde{Y}^{(n)} \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}$ for each $n \geqslant 0$. □

That we need to ask more from Y when we ask less from $\langle M\rangle$ is a well known fact in Stieltjes integration theory where this is often used to trade regularity of the integrand against the regularity of the integrating function. Still, one is left to wonder what is $\overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}$ when $t \mapsto \langle M \rangle_t$ is not absolutely continuous. A criterion is offered in:

Lemma 4.3 *Let $M \in \mathscr{M}_2^{\rm cont}$ and $Y \in \mathcal{V}_M$. Then*

(1) $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}$
(2) $\exists \widetilde{Y} \in \mathcal{V}_M$ *progressively measurable such that*

$$\int_0^\infty 1_{\{Y_t \neq \widetilde{Y}_t\}} \mathrm{d}\langle M \rangle_t = 0 \quad \text{a.s.} \tag{4.17}$$

are equivalent.

Note that the map $t, \omega \mapsto 1_{\{Y_t(\omega) \neq \widetilde{Y}_t(\omega)\}}$ is measurable and so the Lebesgue-Stieltjes integral (4.17) is well defined and defines a random variable. We leave the proof of Lemma 4.3 to a homework assignment.

Examples of processes that are measurable, adapted, but not progressively measurable exist although they seem to invariably capitalize on a huge difference between $\mathcal{F}$ (which determines measurability) and $\mathcal{F}_t$ (which determines progressive measurability). Still, assuming progressive measurability from the outset is not a considerable loss because of the following result:

Theorem 4.4 *Given a probability space $(\Omega, \mathcal{F}, P)$ and a jointly measurable process Y, for each filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$ there exists a progressively measurable process $\widetilde{Y}$ that is a version of Y in the sense that $\forall t \geqslant 0\colon P(Y_t \neq \widetilde{Y}_t) = 0$.*

Proof. This can found in several advanced texts albeit, apparently, with difficult proofs. A simple proof has appeared recently in M. Onderját and J. Seilder's paper "On existence of progressively measurable modifictions" published in Electronic Communications in Probability, vol. 18, no. 20, year 2013, pages 1-6. The key step of that proof is to show that, given any $B \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$, the process $\{E(1_B(t, \cdot) | \mathcal{F}_t) \colon t \geqslant 0\}$ admits a progressively measurable version; the rest are approximation arguments. The proof works in great generality; namely, for Y taking values in any Polish space. $\square$

Since (by Tonnelli), any two versions Y and $\widetilde{Y}$ of the same process necessarily agree away from a Lebesgue null-set of times a.s., the criterion (4.17) holds for any $M \in \mathscr{M}_2^{\rm cont}$ with absolutely continuous $\langle M \rangle$, thus reproducing the conclusion of Proposition 4.1 from Proposition 4.2 and Theorem 4.4.

4.2 Localized Itô integral.

The restriction of the above stochastic integral to square integrable processes and square integrable integrands is at times too restrictive. Fortunately, there is a way to avoid it by a procedure referred to as *localization* which amounts to "stopping" the processes at conveniently defined stopping times that effectively revert the integration to the square integrable case.

Given $M \in \mathscr{M}_{\text{loc}}^{\text{cont}}$, let

$$\mathcal{V}_M^{\text{loc}} := \left\{ Y \colon \text{measurable} \,\wedge\, \text{adapted} \,\wedge\, \forall t \geqslant 0 \colon \int_0^t Y_s^2 \mathrm{d}\langle M\rangle_s < \infty \text{ a.s.} \right\} \tag{4.18}$$

be the class of processes that are *locally integrable* with respect to M. We then have:

Theorem 4.5 *Let $M \in \mathscr{M}_{\text{loc}}^{\text{cont}}$ and $Y \in \mathcal{V}_M^{\text{loc}}$. For each $K \geqslant 0$ set*

$$\tau_K := \inf\left\{ t \geqslant 0 \colon \langle M\rangle_t \geqslant K \,\vee\, \int_0^t Y_s^2 \mathrm{d}\langle M\rangle_t \geqslant t \right\} \tag{4.19}$$

and denote $M_t^{(K)} := M_{t\wedge\tau_K}$. Then $M^{(K)} \in \mathscr{M}_2^{\text{cont}}$ and $Y \in \mathcal{V}_{M^{(K)}}$. Moreover,

$$\forall \widetilde{K} \geqslant K > 0 \;\forall t \geqslant 0 \colon \quad \int_0^t Y_s \,\mathrm{d}M_s^{(\widetilde{K})} = \int_0^t Y_s \,\mathrm{d}M_s^{(K)} \quad \text{a.s. on } \{\tau_K > t\} \tag{4.20}$$

and so

$$\forall t \geqslant 0 \colon \quad \int_0^t Y_s \,\mathrm{d}M_s := \lim_{K\to\infty} \int_0^t Y_s \,\mathrm{d}M_s^{(K)} \;\text{ exists a.s.} \tag{4.21}$$

Moreover, assuming $\mathcal{F}_0$ contains all P-null sets, the process $\{\int_0^t Y_s \mathrm{d}M_s \colon t \geqslant 0\}$ admits a continuous version $I(Y) := \{I_t(Y) \colon t \geqslant 0\}$ which is a continuous local martingale with

$$\forall t \geqslant 0 \colon \quad \big\langle I(Y)\big\rangle_t = \int_0^t Y_s^2 \mathrm{d}\langle M\rangle_s \quad \text{a.s.} \tag{4.22}$$

Finally, for any stopping time T and any $t \geqslant 0$,

$$\int_0^{T\wedge t} Y_s \mathrm{d}M_s := I_{T\wedge t}(Y) = \int_0^t Y_s 1_{\{T>s\}} \mathrm{d}M_s \quad \text{a.s.} \tag{4.23}$$

where the integral on the right is in the sense (4.21).

Since the proof is almost exactly the same as for the Brownian case, we proceed only by some remarks. Note that in this case we need to truncate both the quadratic variation of the integral to be defined and that of the underlying martingale. This was not needed when M was standard Brownian motion because then $\langle M\rangle$ was explicit. Also note that for Brownian motion we truncated the integrals slightly differently; namely, by writing $\int_0^t Y_s 1_{\{\tau_K>s\}} \,\mathrm{d}M_s$ instead of $\int_0^t Y_s \,\mathrm{d}M_s^{(K)}$. The reason is that the former integral is still only in the localized sense while the latter is a proper L^2-Itô integral. That these are the same follows from (4.23).

Further reading: Sections 3.1-3.3 of Karatzas-Shreve