## 3. Itô integral w.r.t. continuous $L^2$-martingales

In 275D we defined the stochastic, a.k.a. Itô, integral $\int_0^t Y_s \mathrm{d}B_s$ with respect to the standard Brownian motion $B$. With continuous martingales under control, we will now move to extending this definition to $B$ replaced by any continuous local martingale. For convenience we will henceforth denote

$$\begin{aligned}
\mathscr{M} &:= \{M\colon \text{martingale}\} \\
\mathscr{M}_2 &:= \{M\colon L^2\text{-martingale}\} \\
\mathscr{M}^{\text{cont}} &:= \{M\colon \text{continuous martingale}\} \\
\mathscr{M}_2^{\text{cont}} &:= \{M\colon \text{continuous } L^2\text{-martingale}\} \\
\mathscr{M}_{\text{loc}}^{\text{cont}} &:= \{M\colon \text{continuous local martingale}\}
\end{aligned} \tag{3.1}$$

with these over a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ that will both be clear from context.

The construction of the stochastic integral with respect to a continuous martingale proceeds very much as for the stochastic integral with respect to standard Brownian motion, so we will be somewhat terse on detail and motivation in what follows. We start with:

**Definition 3.1** (Simple process) *A stochastic process $Y$ on a probability space $(\Omega, \mathcal{F}, P)$ endowed with filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ is said to be simple if there exist $n \geqslant 0$, times $0 = t_0 < t_1 < \ldots, t_n$ and random variables $Z_0, \ldots, Z_n \in L^\infty$ such that $Z_i$ is $\mathcal{F}_{t_{i-1}\vee 0}$-measurable for each $i = 0, \ldots, n$ and*

$$Y_t = Z_0 1_{\{0\}}(t) + \sum_{i=1}^n Z_i 1_{(t_{i-1}, t_i]}(t) \tag{3.2}$$

*holds for each $t \geqslant 0$. We write $\mathcal{V}_0 := \{Y\colon \text{ simple}\}$.*

Given a simple process $\{Y_t\colon t \geqslant 0\}$ with representation (3.2) and another stochastic process $\{M_t\colon t \geqslant 0\}$, for each $t \geqslant 0$ we define

$$\int_0^t Y_s \mathrm{d}M_s := \sum_{i=1}^n Z_i (M_{t\wedge t_i} - M_{t\wedge t_{i-1}}). \tag{3.3}$$

Using the same argument as for the Riemann sums, the resulting object $\int_0^t Y_s \mathrm{d}M_s$ does not depend on the representation (3.2) of $Y$. The resulting integral then obeys:

**Lemma 3.2** *Let $Y \in \mathcal{V}_0$ and $M \in \mathscr{M}_2^{\text{cont}}$. Then*

$$\left\{\int_0^t Y_s \mathrm{d}M_s\colon t \geqslant 0\right\} \in \mathscr{M}_2^{\text{cont}} \tag{3.4}$$

*and the Itô isometry*

$$E\left[\left(\int_0^t Y_s \mathrm{d}M_s\right)^2\right] = E\left[\int_0^t Y_s^2 \mathrm{d}\langle M\rangle_s\right] \tag{3.5}$$

*holds true with the integral on the right formally in Lebesgue-Stieltjes sense (the integral is actually a Riemann sum). In fact, we even have*

$$\Big\langle \int_0^{\cdot} Y_s \mathrm{d}M_s \Big\rangle_t = \int_0^t Y_s^2 \mathrm{d}\langle M\rangle_s \tag{3.6}$$

*for each $t \geqslant 0$.*

*Proof.* Being a finite sum of terms of the form $Z_i(M_{t\wedge t_i} - M_{t_{i-1}\wedge t})$, where $Z_i$ is bounded and $M_{t\wedge t_i} - M_{t_{i-1}\wedge t} \in L^2$, the integral is in $L^2$ for each $t \geqslant 0$.

Next let $0 \leqslant u \leqslant t$ and assume $Y$ is represented so that $u$ is one of the partition points in $\{t_i\}_{i=0}^n$, say, $u = t_i$. Then for all $j > i$,

$$E\big[Z_i(M_{t\wedge t_j} - M_{t\wedge t_{j-1}})\,\big|\mathcal{F}_u\big] = 0 \quad \text{a.s.} \tag{3.7}$$

and so

$$E\bigg[\int_0^t Y_s \mathrm{d}M_s \,\bigg|\, \mathcal{F}_u\bigg] = \int_0^u Y_s \mathrm{d}M_s \quad \text{a.s.} \tag{3.8}$$

proving that the integral is an $L^2$-martingale. The continuity is obvious from the continuity of $M$. For the second moment the martingale property shows that, almost surely,

$$E\bigg[\Big(\int_0^t Y_s \mathrm{d}M_s\Big)^2 \,\bigg|\, \mathcal{F}_u\bigg] = \Big(\int_0^t Y_s \mathrm{d}M_s\Big)^2 + E\bigg[\Big(\sum_{j=i+1}^n Z_i(M_{t\wedge t_j} - M_{t\wedge t_{j-1}})\Big)^2 \,\bigg|\, \mathcal{F}_u\bigg]. \tag{3.9}$$

The second term is now computed using the fact that the increments of $M$ are conditionally uncorrelated with $E((M_t - M_s)^2|\mathcal{F}_u) = E(\langle M\rangle_t - \langle M\rangle_s|\mathcal{F}_u)$ whenever $u \leqslant s \leqslant t$ by the fact that $\{M_t^2 - \langle M\rangle_t \colon t \geqslant 0\} \in \mathscr{M}_2$ to get

$$\begin{aligned}
\sum_{j,k=i+1}^n E\Big(Z_j Z_k(M_{t\wedge t_j} - M_{t\wedge t_{j-1}})(M_{t\wedge t_k} - M_{t\wedge t_{k-1}}) \,\Big|\, \mathcal{F}_u\Big)& \\
= \sum_{j=i+1}^n E\Big(Z_j^2(M_{t\wedge t_j} - M_{t\wedge t_{j-1}})^2 \,\Big|\, \mathcal{F}_u\Big)& \\
= \sum_{j=i+1}^n E\Big(Z_j^2(\langle M\rangle_{t\wedge t_j} - \langle M\rangle_{t\wedge t_{j-1}}) \,\Big|\, \mathcal{F}_u\Big)& \\
= E\bigg(\int_u^t Y_s^2 \mathrm{d}\langle M\rangle_s \,\bigg|\, \mathcal{F}_u\bigg) \quad \text{a.s.}&
\end{aligned} \tag{3.10}$$

Plugging this in (3.9) proves both (3.6) and, by taking expectations, also (3.5). □

The Itô isometry can be thought of as a statement of uniform continuity of the map $Y \mapsto \int_0^t Y_s \mathrm{d}M_s$ with respect to the pseudometric $Y, \widetilde{Y} \mapsto [\![Y - \widetilde{Y}]\!]_M$ where

$$[\![Y]\!]_M := \sum_{n\geqslant 1} 2^{-n} \min\bigg\{1,\, E\bigg(\int_0^n Y_s^2 \mathrm{d}\langle M\rangle_s\bigg)\bigg\}, \tag{3.11}$$

where the subindex $M$ reminds us that this concept depends sensitively on $M$. The expression is meaningful for each $Y \in \mathcal{V}_M$ where

$$\mathcal{V}_M := \left\{ Y\colon \text{adapted} \,\wedge\, \text{measurable} \,\wedge\, \forall t \geqslant 0 : E\left( \int_0^t Y_s^2 \mathrm{d}\langle M\rangle_s \right) < \infty \right\}. \tag{3.12}$$

We could endow $\mathcal{V}_M$ with the structure of a topological vector space by identifying processes modulo equivalence on a $\langle M\rangle$-null set of times, but this is not really required to carry out the arguments we need to make. So, instead, we will treat it as a set of processes whose "distances" are measured using the pseudometric supplied by $[\![\cdot]\!]_M$.

Note that $\mathcal{V}_0 \subseteq \mathcal{V}_M$ and, in light of the remarks we just made, set

$$\overline{\mathcal{V}_0}^{[\![\cdot]\!]_M} := \left\{ Y \in \mathcal{V}_M\colon \left( \exists \{Y^{(n)}\}_{n\in\mathbb{N}} \in \mathcal{V}_0^{\mathbb{N}}\colon \lim_{n\to\infty} [\![Y - Y^{(n)}]\!]_M = 0 \right) \right\} \tag{3.13}$$

The extension of the Itô integral to more general processes is then stated formally as:

**Lemma 3.3** *Let $M \in \mathscr{M}_2^{\mathrm{cont}}$ and $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}$. Then there exists a family*

$$\left\{ \int_0^t Y_s \mathrm{d}M_s\colon t \geqslant 0 \right\} \tag{3.14}$$

*of random variables such that for each $\{Y^{(n)}\}_{n\in\mathbb{N}} \in \mathcal{V}_0^{\mathbb{N}}$,*

$$[\![Y - Y^{(n)}]\!]_M \underset{n\to\infty}{\longrightarrow} 0 \;\;\Rightarrow\;\; \forall t \geqslant 0\colon\; \int_0^t Y_s^{(n)} \mathrm{d}M_s \xrightarrow[n\to\infty]{L^2} \int_0^t Y_s \mathrm{d}M_s \tag{3.15}$$

*Moreover,*

$$E\left[ \left( \int_0^t Y_s \mathrm{d}M_s \right)^2 \right] = E\left[ \int_0^t Y_s^2 \mathrm{d}\langle M\rangle_s \right] \tag{3.16}$$

*holds for each $t \geqslant 0$. The process* (3.14) *is an $L^2$ martingale.*

*Proof.* Since $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}$, there exists at least one $\{Y^{(n)}\}_{n\in\mathbb{N}} \in \mathcal{V}_0^{\mathbb{N}}$ with $[\![Y - Y^{(n)}]\!]_M \to 0$. For any such sequence, each $t \geqslant 0$ and $m, n \geqslant 0$, the Itô isometry gives

$$E\left[ \left| \int_0^t Y_s^{(n)} \mathrm{d}M_s - \int_0^t Y_s^{(m)} \mathrm{d}M_s \right|^2 \right] = E\left[ \int_0^t (Y_s^{(n)} - Y_s^{(m)})^2 \mathrm{d}\langle M\rangle_s \right] \tag{3.17}$$

By $[\![Y - Y^{(n)}]\!]_M \to 0$ the right-hand side tends to zero as $m, n \to \infty$. It follows that $\{\int_0^t Y_s^{(n)} \mathrm{d}M_s\}_{n\in\mathbb{N}}$ is Cauchy in $L^2$ of the probability space and thus converges to a random variable that we denote $\int_0^t Y_s \mathrm{d}M_s$. This random variable is determined only up to changes on a null set. Still, standard facts about $L^2$ convergence ensure that it serves as the $L^2$-limit for all sequences $\{Y^{(n)}\}_{n\in\mathbb{N}} \in \mathcal{V}_0^{\mathbb{N}}$ with $[\![Y - Y^{(n)}]\!]_M \to 0$. The Itô formula (3.16) then extends from (3.5) by the fact that $L^2$-convergence implies convergence of the associated $L^2$-norms. That (3.14) is a martingale follows by the fact that $L^2$-convergence preserves the martingale property. □

The last item to address is the regularity of the process (3.14). Here we get:

**Lemma 3.4** *Let $M \in \mathscr{M}_2^{\rm cont}$ and $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]_M}$. Then the process* (3.14) *admits a continuous version $\{I_t(Y)\colon t \geqslant 0\}$ which, assuming that $\mathcal{F}_0$ contains all $P$-null sets, is a continuous $L^2$-martingale with quadratic variation determined by*

$$\forall t \geqslant 0\colon\ \big\langle I(Y)\big\rangle_t = \int_0^t Y_s^2 \mathrm{d}\langle M\rangle_s \quad \text{a.s.} \tag{3.18}$$

*The integral on the right is in Lebesgue-Stieltjes sense (recall that $Y$ is jointly measurable).*

*Proof.* Let $\{Y^{(n)}\}_{n\in\mathbb{N}} \in \mathcal{V}_0^{\mathbb{N}}$ with $[\![Y - Y^{(n)}]\!]_M \to 0$ and abbreviate $I_t(Y^{(n)}) := \int_0^u Y_s^{(n)} \mathrm{d}M_s$. Using that the Itô integral of a process in $\mathcal{V}_0$ is a continuous $L^2$-martingale, the $L^2$-Doob inequality upgrades (3.17) to the bound

$$\begin{aligned} &E\Big(\sup_{u\leqslant t}\big|I_u(Y^{(n)}) - I_u(Y^{(n+1)})\big|^2\Big) \\ &\qquad \leqslant 4E\Big(\big|I_t(Y^{(n)}) - I_t(Y^{(n+1)})\big|^2\Big) \leqslant 4E\bigg(\int_0^t (Y_s^{(n)} - Y_s^{(n+1)})^2 \mathrm{d}\langle M\rangle_s\bigg). \end{aligned} \tag{3.19}$$

For $t := n$, we either have $4^n[\![Y^{(n)} - Y^{(n+1)}]\!]_M^2 \geqslant 1$ or the expectation on the right is less than $4^n[\![Y^{(n)} - Y^{(n+1)}]\!]_M^2$. Chebyshev inequality then gives

$$P\Big(\sup_{u\leqslant n}\big|I_u(Y^{(n)}) - I_u(Y^{(n+1)})\big| > 2^{-n}\Big) \leqslant 4\cdot 4^n \cdot 4^n [\![Y^{(n)} - Y^{(n+1)}]\!]_M^2 \tag{3.20}$$

Now assume that $[\![Y - Y^{(n)}]\!]_M \leqslant 32^{-n}$. We then have $[\![Y^{(n)} - Y^{(n+1)}]\!]_M^2 \leqslant 4\cdot 32^{-n}$ by the triangle inequality and the probability is summable on $n$. Define

$$\Omega_0 := \Omega \smallsetminus \Big\{\sup_{u\leqslant n}\big|I_u(Y^{(n)}) - I_u(Y^{(n+1)})\big| > 2^{-n} \text{ i.o.}\Big\} \tag{3.21}$$

The Borel-Cantelli lemma shows $P(\Omega_0) = 1$. For each $t \geqslant 0$ define

$$I_t(Y) := \begin{cases} \lim_{n\to\infty} I_t(Y^{(n)}), & \text{on } \Omega_0, \\ 0, & \text{else,} \end{cases} \tag{3.22}$$

where the limit exists in light of the fact that the sequence $\{I_t(Y^{(n)}) - I_t(Y^{(n+1)})\}_{n\in\mathbb{N}}$ is absolutely summable on $\Omega_0$. Since the summability is locally uniform in $t$, the fact that each $t \mapsto I_t(Y^{(n)})$ is continuous implies that $t \mapsto I_t(Y)$ is continuous.

Lemma 3.3 implies that $\{I_t(Y)\colon t \geqslant 0\}$ is a version of (3.14). The pointwise convergence along with the fact that $\Omega_0 \in \mathcal{F}_0$ shows that the process is adapted and so it is a continuous martingale. In order to prove (3.18), let $u \leqslant t$ and $A \in \mathcal{F}_u$. Then (3.6) along with the fact that $\{I_t^{(n)} \colon t \geqslant 0\}$ and $\{I_t(Y) \colon t \geqslant 0\}$ are $L^2$-martingales give

$$\begin{aligned}
E\Big(1_A\big(I_t(Y)^2 - I_s(Y)^2\big)\Big) &= E\Big(1_A\big(I_t(Y) - I_s(Y)\big)^2\Big) \\
&= \lim_{n\to\infty} E\Big(1_A(I_t(Y^{(n)}) - I_s(Y^{(n)}))^2\Big) = \lim_{n\to\infty} E\Big(1_A\big((I_t(Y^{(n)}))^2 - (I_s(Y^{(n)}))^2\big)\Big) \\
&= \lim_{n\to\infty} E\left(1_A\left(\int_0^t (Y_s^{(n)})^2 \mathrm{d}\langle M\rangle_s - \int_0^t (Y_s^{(n)})^2 \mathrm{d}\langle M\rangle_s\right)\right) \\
&= E\left(1_A\left(\int_0^t Y_s^2 \mathrm{d}\langle M\rangle_s - \int_0^t Y_s^2 \mathrm{d}\langle M\rangle_s\right)\right),
\end{aligned} \tag{3.23}$$

where we again called on the $L^1$ convergence $\int_0^t (Y_s^{(n)})^2 \mathrm{d}\langle M\rangle_s \to \int_0^t Y_s^2 \mathrm{d}\langle M\rangle_s$ implied by $[\![Y - Y^{(n)}]\!]_M \to 0$. As this holds for all $A \in \mathcal{F}_s$, it follows that $I_t(Y)^2 - \int_0^t Y_s^2 \mathrm{d}\langle M\rangle_s$ is a martingale, proving (3.18). $\square$

Further reading: Sections 3.1-3.2ABC of Karatzas-Shreve