25. Picard's theorem and beyond

We proceed by stating another theorem concerning existence and, this time also uniqueness, of the solution to above ODE. Then we note an example that shows why the stated treatment fails when the driving signal is finite p-variation for $p = 2$ but not any $p < 2$.

25.1 Generalized Picard theorem.

In treatments of standard ODEs, uniqueness is typically addressed by imposing the condition that the function h is Lipschitz continuous. (Note that uniqueness fails for one-dimensional ODEs whenever $y \mapsto 1/h(y)$ is locally integrable near a zero of h, so Lipschitz continuity also nearly necessary.) This also leads to a different proof of existence, based on convergence of so called *Picard-Lindelöf iterations* defined recursively by setting, for each $t \geqslant 0$,

$$y_0(t) := y_0 \tag{25.1}$$

and letting

$$y_{n+1}(t) := y_0 + \int_0^t h(y_n)\mathrm{d}x \tag{25.2}$$

for each $n \in \mathbb{N}$. One advantage of this approach is control of rate of convergence of the iterates, and thus possibility to approximate the unique solution up to a known error. In addition, one also gets continuity of the solution in the initial value y_0 and the driving signal. (This would be a meaningless statement without uniqueness.)

The classical Picard theorem does the above for $x(t) := t$ and h Lipschitz continuous. Our generalization to rougher signals requires just a bit more regularity than that; namely, that h is differentiable with h' Hölder continuous:

Theorem 25.1 (Generalized Picard theorem) *Let $h\colon \mathbb{R} \to \mathbb{R}$ be differentiable with h' locally α-Hölder continuous for some $\alpha \in (0,1]$. Let $x\colon \mathbb{R}_+ \to \mathbb{R}$ be such that $\forall t > 0\colon x \in V^p([0,t])$ for some $p < 1+\alpha$. For each $y_0 \in \mathbb{R}$, there exists $T > 0$ and a unique $y \in V^p([0,T])$ such that*

$$\forall t \in [0,T]\colon\ \ y(t) = y_0 + \int_0^t h(y)\mathrm{d}x \tag{25.3}$$

Moreover, $y_0, x \mapsto y$ is continuous as a map $\mathbb{R} \times V^p([0,T]) \to V^p([0,T])$.

The assumption of Hölder continuity of the derivative enters through a rather involved estimate stated and proved in:

Lemma 25.2 *Let $h\colon \mathbb{R} \to \mathbb{R}$ be differentiable with h' such that $|h'(z) - h'(\tilde z)| \leqslant K|z - \tilde z|^\alpha$ for each $z, \tilde z \in \mathbb{R}$. Suppose that $I \subseteq \mathbb{R}$ is a bounded interval and $y, \tilde y \in \mathcal{V}^p(I)$ for some $p \geqslant 1$. Then*

$$\|h \circ y - h \circ \tilde y\|_{\mathcal{V}^{p/\alpha}(I)} \leqslant \Bigl(2\|h'\| + 2K\bigl(\|y\|_{p,I}^\alpha + \|\tilde y\|_{p,I}^\alpha\bigr)\Bigr)\|y - \tilde y\|_{\mathcal{V}^p(I)} \tag{25.4}$$

where $\|h'\| := \sup\{|h'(z)|\colon \|z\| \leqslant r\}$ for $r := \sup_{t\in I} \max\{\|y(t)\|, \|\tilde y(t)\|\}$.

Proof. The Fundamental Theorem of Calculus gives us $h(z) - h(\tilde z) = g(z,\tilde z)(z - \tilde z)$ for $g(z,\tilde z) := \int_0^1 h'(sz + (1-s)\tilde z)\mathrm{d}s$. Note that we have

$$\bigl|g(z,\tilde z)\bigr| \leqslant \|h'\| \tag{25.5}$$

and

$$\begin{aligned}\big|g(z,\tilde z)-g(z',\tilde z')\big|^{p/\alpha}&\leqslant\left(K\int_0^1\mathrm{d}s\big(s|z-z'|+(1-s)|\tilde z-\tilde z'|\big)^{\alpha}\right)^{p/\alpha}\\&\leqslant K^{p/\alpha}\big(|z-z'|^p+|\tilde z-\tilde z'|^p\big)\end{aligned}\tag{25.6}$$

Given any $s,t\in I$, the rewrite

$$\begin{aligned}&h(y_t)-h(\tilde y_t)-\big(h(y_s)-h(\tilde y_s)\big)\\&\qquad=g(y_t,\tilde y_t)\big(y_t-\tilde y_t-(y_s-\tilde y_s)\big)+\big(g(y_t,\tilde y_t)-g(y_s,\tilde y_s)\big)(y_s-\tilde y_s)\end{aligned}\tag{25.7}$$

along with the inequality $(a+b)^\gamma\leqslant 2^{\gamma-1}(a^\gamma+b^\gamma)$ for any $a,b\geqslant 0$ and $\gamma\geqslant 1$ give

$$\begin{aligned}&\Big|h(y_t)-h(\tilde y_t)-\big(h(y_s)-h(\tilde y_s)\big)\Big|^{p/\alpha}\\&\qquad\leqslant 2^{p/\alpha-1}\|h'\|^{p/\alpha}\Big|y_t-\tilde y_t-(y_s-\tilde y_s)\Big|^{p/\alpha}\\&\qquad\qquad+2^{p/\alpha-1}K^{p/\alpha}\big(|y_t-y_s|^p+|\tilde y_t-\tilde y_s|^p\big)|y_s-\tilde y_s|^{p/\alpha}\end{aligned}\tag{25.8}$$

Bounding the very last factor by $|y_s-\tilde y_s|^{p/\alpha}\leqslant\|y-\tilde y\|_{\infty,I}$ and using the resulting inequality over intervals in a partition Π and optimizing over the partition gives

$$\begin{aligned}&V^{p/\alpha}(h\circ y-h\circ\tilde y,I)\\&\qquad\leqslant 2^{p/\alpha-1}\|h'\|^{p/\alpha}V^{p/\alpha}(y-\tilde y,I)\\&\qquad\qquad+2^{p/\alpha-1}K^{p/\alpha}\big(V^p(y,I)+V^p(\tilde y,I)\big)\|y-\tilde y\|_{\infty,I}\end{aligned}\tag{25.9}$$

Taking α/p-power with the help of the inequality $(a+b)^{\alpha/p}\leqslant a^{\alpha/p}+b^{\alpha/p}$ shows

$$\begin{aligned}&\|h\circ y-h\circ\tilde y\|_{p/\alpha,I}\\&\qquad\leqslant 2^{1-\alpha/p}\|h'\|\|y-\tilde y\|_{p/\alpha,I}+2^{1-\alpha/p}K\big(\|y\|_{p,I}^\alpha+\|\tilde y\|_{p,I}^\alpha\big)\|y-\tilde y\|_{\infty,I}\end{aligned}\tag{25.10}$$

Bounding $2^{1-\alpha/p}\leqslant 2$ and combining this with

$$\|h\circ y-h\circ\tilde y\|_{\infty,I}\leqslant\|h'\|\|y-\tilde y\|_{\infty,I}\tag{25.11}$$

the claim follows from downward monotonicity of $p\mapsto\|f\|_{p,I}$. $\square$

We are now ready to give:

Proof of Theorem 25.1. Suppose $h\colon\mathbb{R}\to\mathbb{R}$ is differentiable with $|h'(z)-h'(\tilde z)|\leqslant K|z-\tilde z|^\alpha$ for each $z,\tilde z\in\mathbb{R}$. Let I be an interval of the form $[0,t]$ and let $y,\tilde y\in V^p(I)$ be such that $y(0)=\tilde y(0)=y_0$. Note that $F(y)(t):=y_0+\int_0^t h\circ y\,\mathrm{d}x$ is well defined for each $t\in I$ by the fact that $h\circ y\in V^p(I)$, and similarly for $F(\tilde y)$. Fix $r>0$ and suppose that $\max\{\|y(t)\|,\|\tilde y(t)\|\}\leqslant r$ for all $t\in I$. Then

$$\begin{aligned}\max\big\{\|F(y)-y_0\|_{\infty,I},\|F(y)\|_{p,I}\big\}&\leqslant\|h\circ y\|_{\infty,I}\|x\|_{p,I}+C_{p,p}\|h\circ y\|_{p,I}\|x\|_{p,I}\\&\leqslant\Big(\|h(y_0)\|+2C_{p,p}\|h'\|\|y\|_{p,I}\Big)\|x\|_{p,I}\end{aligned}\tag{25.12}$$

where $\|h'\| := \sup\{|h'(z)|\colon \|z\| \leqslant r\}$ for $r := \sup_{t\in I}\|y(t)\|$. Similarly, using Lemma 8.2 we also get

$$\begin{aligned}\|F(y)-F(\tilde{y})\|_{V^p(I)} &\leqslant 2C_{p,p/\alpha}\|h\circ y-h\circ\tilde{y}\|_{p/\alpha,I}\|x\|_{p,I}\\ &\leqslant 2C_{p,p/\alpha}\Big(2\|h'\|+2K\big(\|y\|_{p,I}^{\alpha}+\|\tilde{y}\|_{p,I}^{\alpha}\big)\Big)\|y-\tilde{y}\|_{V^p(I)}\|x\|_{p,I}.\end{aligned} \tag{25.13}$$

Now pick $T>0$ so small that, for $J_T := [0,T]$,

$$\Big(\|h(y_0)\|+2C_{p,p}\|h'\|r\Big)\|x\|_{p,J_T} \leqslant r \tag{25.14}$$

and

$$2C_{p,p/\alpha}\big(2\|h'\|+4Kr^{\alpha}\big)\|x\|_{p,J_T} \leqslant \frac{1}{2} \tag{25.15}$$

The bound (25.12) then shows that F maps

$$\mathcal{K}_T := \Big\{y\in V^p(J_T)\colon y(0)=y_0 \,\wedge\, \|y\|_{V^p(J_T)}\leqslant r\Big\} \tag{25.16}$$

into itself while (25.13) shows that F is a contraction on $\mathcal{K}_T$, i.e.,

$$\forall y,\tilde{y}\in\mathcal{K}_T\colon\;\; \|F(y)-F(\tilde{y})\|_{V^p(J_T)} \leqslant \frac{1}{2}\|y-\tilde{y}\|_{V^p(J_T)} \tag{25.17}$$

Since $\mathcal{K}_T$ is also closed in $V^p(J_T)$, the Banach fixed point theorem implies existence of a unique $y\in\mathcal{K}_T$ such that $y=F(y)$. By the fact that every solution of the ODE will lie in $\mathcal{K}_{T'}$ for some $T'>0$ small, also the solution is locally unique. Applying this inductively, the solution is thus unique globally as well.

It remain to prove continuity in the initial data and the driving signal. For simplicity, we will only prove the continuity in x. Let $\phi\colon V^p(J_T)\to V^p(J_t)$ denote the solution map $x\mapsto\phi(x)$. Given two signals $x,\tilde{x}\in V^p(J_T)$ such that, without loss of generality, $\|\tilde{x}\|_{p,J_T}\leqslant\|x\|_{p,J_T}$, note that

$$\phi_t(x)-\phi_t(\tilde{x}) = \int_0^t\big[h\circ\phi(x)-h\circ\phi(\tilde{x})\big]\mathrm{d}x+\int_0^t h\circ\phi(\tilde{x})\mathrm{d}(x-\tilde{x}) \tag{25.18}$$

Interpreting the first integral as $F(\phi(x))-F(\phi(\tilde{x}))$ for F defined using x, the contractivity of F in (25.17) gives

$$\|\phi(x)-\phi(\tilde{x})\|_{V^p(J_T)} \leqslant \frac{1}{2}\|\phi(x)-\phi(\tilde{x})\|_{V^p(J_T)}+\|h\circ\phi(\tilde{x})\|_{V^p(J_T)}\|x-\tilde{x}\|_{p,I} \tag{25.19}$$

and so, noting that

$$\|h\circ\phi(\tilde{x})\|_{V^p(J_T)} \leqslant \Big(\|h(y_0)\|+2C_{p,p}\|h'\|\|\phi(\tilde{x})\|_{p,I}\Big) \tag{25.20}$$

we get

$$\|\phi(x)-\phi(\tilde{x})\|_{V^p(J_T)} \leqslant 2r\frac{\|x-\tilde{x}\|_{p,J_T}}{\|x\|_{p,J_T}} \tag{25.21}$$

by invoking (25.14). This is the desired statement of continuity in the driving signal. □

25.2 What happens for $p \geqslant 2$?.

With the above results proving good control of the realm of processes in $V^p([0,T])$ with $p < 2$, a natural question to ask is: What can be done when the driving signal of the ODE lies in $V^p([0,T])$ for $p \geqslant 2$ but not for any $p < 2$? A simple answer to this is: not much, because pretty much everything goes wrong in this case.

The primary obstacle to overcome is the definition of the integral. Here the traditional definition of the Riemann-Stieltjes integral does run into problems due to the following fact: Given a compact interval I, let $\Pi = (\{t_i\}_{i=0}^n, \{t_{i-1}\}_{i=1}^n)$ and $\Pi = (\{t_i\}_{i=0}^n, \{t_i\}_{i=1}^n)$ be two partitions of I with the same partition points, but the left-endpoint marked points for Π and the right-endpoint marked points for Π'. Then

$$S(f,f,\Pi') - S(f,f,\Pi) = V^2(f,I,\Pi) \tag{25.22}$$

and so once f is only finite second variation, one should not expect that the Riemann-Stieljes sum becomes independent of the marked points when the mesh of the partition tends to zero. (That this is in fact a problem can be checked by setting f to be a path of standard Brownian motion which is in $V^p(I)$ a.s. for each $p > 2$ but for which $V^2(f,I,\Pi_n) \to |I|$ a.s. if $\|\Pi_n\| \to 0$ so fast that $\sum_{n\geqslant 1} \|\Pi_n\| < \infty$.)

Another approach one might want to try is to define integral directly by functional analytic means. Unfortunately, also this fails rather spectacularly:

Lemma 25.3 *Let $p \geqslant 2$ and let $I \subseteq \mathbb{R}$ be a non-degenerate compact interval. Then the bilinear map $f, g \mapsto \int f \mathrm{d}g$, defined on smooth functions as the ordinary Stietjes integral, admits no continuous extension to $V^p(I) \times V^p(I)$. In fact, the map is not even bounded in $\|\cdot\|_{p,I}$-norm.*

Proof. We will prove the second part of the statement as that implies the first. By shift and scaling it suffices to prove this for $I := [0, 2\pi]$. Given $b > 1$ and $\alpha > 0$, consider the following functions

$$f_n(t) := \sum_{k=1}^{n} \frac{1}{\sqrt{k}} b^{-\alpha k} \cos(b^n t) \tag{25.23}$$

and

$$g_n(t) := \sum_{k=1}^{n} \frac{1}{\sqrt{k}} b^{-\alpha k} \sin(b^n t) \tag{25.24}$$

These converge pointwise to

$$f(t) := \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} b^{-\alpha k} \cos(b^n t) \tag{25.25}$$

and

$$g(t) := \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} b^{-\alpha k} \sin(b^n t), \tag{25.26}$$

respectively.

Next assume $\alpha \in (0,1)$. We claim that $f_n \to f$ and $g_n \to g$ in $V^{1/\alpha}(I)$. For this we proceed similarly as in the proof of (23.14): Given $s < t$, bound $|\cos(b^k t) - \cos(b^k s)|$ by

$b^k|t-s|$ when $k \leqslant \log_b 1/|t-s|$ and by 2 otherwise to get

$$\begin{aligned} \big|(f-f_n)(t)-(f-f_n)(s)\big| &\leqslant \sum_{k=n+1}^{\infty} \frac{1}{\sqrt{k}}\, b^{-\alpha k}|\cos(b^k t)-\cos(b^k s)| \\ &\leqslant \frac{1}{\sqrt{n+1}}\Bigg(\sum_{k\leqslant \log_b 1/|t-s|} b^{(1-\alpha)k}|t-s| + 2\sum_{k>\log_b 1/|t-s|} b^{-k\alpha}\Bigg) \\ &\leqslant \frac{K}{\sqrt{n+1}}|t-s|^{\alpha} \end{aligned} \tag{25.27}$$

for K as defined after (23.14). This now shows

$$\|f-f_n\|_{1/\alpha,I} \leqslant \frac{K}{\sqrt{n+1}}|I|^{\alpha} \tag{25.28}$$

proving that $f_n \to f$ in $V^{1/\alpha}(I)$. (The proof of convergence $g_n \to g$ in $V^{1/\alpha}(I)$ is completely analogous so we omit it.)

Next assume that b is an integer with $b \geqslant 2$. Then

$$\begin{aligned} \int_0^{2\pi} f_n \mathrm{d}g_n &= \sum_{k,\ell=1}^{n} \frac{1}{\sqrt{k}}\frac{1}{\sqrt{\ell}}\, b^{-\alpha(k+\ell)} \int_0^{2\pi} \cos(b^k t) b^{\ell}\cos(b^{\ell}t)\mathrm{d}t \\ &= \sum_{k=1}^{n} \frac{1}{k}\, b^{(1-2\alpha)k} \int_0^{2\pi} \cos(b^k t)^2 \mathrm{d}t \\ &= \pi \sum_{k=1}^{n} \frac{1}{k}\, b^{(1-2\alpha)k} \end{aligned} \tag{25.29}$$

For $\alpha := 1/p$ with $p \geqslant 2$ this diverges to infinity despite the fact that, as shown just above, both sequences $\{f_n\}_{n\in\mathbb{Z}}$ and $\{g_n\}_{n\in\mathbb{Z}}$ converge in $V^p([0,2\pi])$. The map $f,g \mapsto \int_0^{2\pi} f\mathrm{d}g$ is thus not even bounded in $\|\cdot\|_{p,I}$-norm, let alone continuous. $\square$

The lack of continuity introduces ambiguity in the interpretation of $\int h(y)\mathrm{d}x$ which undermines the notion of y being a solution to the ODE $\mathrm{d}y = h(y)\mathrm{d}x$. This is exactly where the theory of rough paths (to be discussed next) picks up and makes a significant contribution.