24. Solving moderately rough ODEs: Peano theorem

The Young integral opens up a possibility to prove existence of a solution to an ODE that could not be analyzed within Riemann's and even Lebesgue's integration theory. Here we provide a first such result that, as it turns out, is restricted to driving signals of variation $p < 2$. ODEs with such signals are sometimes referred to as moderately rough.

24.1 Peano's theorem.

Recall the ODE (23.1–23.2). There x and y generally take values in possibly different normed-vector spaces $\mathscr{X}$ and $\mathscr{Y}$ and h and g are functions that make the expressions meaningful. This generality allows us to absorb the second integral into the first and consider integral equations of the form

$$y(t) = y_0 + \int_0^t h(y)\mathrm{d}x \tag{24.1}$$

where $h(y)$ is a bounded linear operator $\mathscr{X} \to \mathscr{Y}$. (Hence h is a map $\mathscr{Y} \to \mathcal{L}(\mathscr{X},\mathscr{Y})$, where $\mathcal{L}(\mathscr{X},\mathscr{Y})$ is the space of bounded linear operators $\mathscr{X} \to \mathscr{Y}$.) We will for simplicity ignore this level of generality in what follows and treat the case when both x and y are real valued and $h(y)$ is thus a multiplication by a scalar. (Still, we will use norm notation on various quantities in the proof to indicate where norms are needed.)

In the ODE literature, any discussion of existence of solution to ODEs usually starts with Peano's theorem, sometimes called also the Cauchy-Peano theorem, proved by Peano in 1886 (incorrect proof) and 1890 (correct proof). We will do the same albeit with a proof that allows for generalizations later:

Theorem 24.1 (Peano's theorem) *Let $h \in C(\mathbb{R})$, $y_0 \in \mathbb{R}$ and $x \in \mathcal{V}^1([0,t])$ for each $t > 0$. Then there exists $T > 0$ and a function $y \in \mathcal{V}^1([0,T])$ such that*

$$\forall t \in [0,T]\colon \quad y(t) = y_0 + \int_0^t h \circ y \,\mathrm{d}x \tag{24.2}$$

Proof. Fix any $r > 0$ and let $K := \sup\{\|h(z)\|\colon \|z - y_0\| \leqslant r\}$, which we assume to be non-zero for otherwise there is nothing to prove. Let $T > 0$ be such that $V^1(x, J_T) \leqslant r/K$ for $J_T := [0,T]$. For any $y \in C(J_T)$ let $F(y)$ be the function defined by

$$F(y)(t) := y_0 + \int_0^t h \circ y \,\mathrm{d}x \tag{24.3}$$

Then $\|y - y_0\|_{\infty,J_T} \leqslant r$ gives

$$\|F(y) - y_0\|_{\infty,J_T} \leqslant KV^1(x, J_T) \leqslant r. \tag{24.4}$$

The fact that $F(y)(t) - F(y)(s) = \int_s^t h \circ y \,\mathrm{d}x$ implies $|F(y)(t) - F(y)(s)| \leqslant KV^1(x,[s,t])$. Hence we get

$$\forall I \subseteq J_T\colon \ \|F(y)\|_{1,I} \leqslant KV^1(x, I) \tag{24.5}$$

The map F thus preserves the set

$$\mathcal{K}_T := \Big\{y \in C(J_T) \cap V^1(J_T)\colon y(0) = y_0 \ \wedge \ \|y - y_0\|_{\infty,J_T} \leqslant r \ \wedge \ \text{(24.5) holds}\Big\} \tag{24.6}$$

This set is bounded and, by the lower-semicontituity of the first variation shown in Lemma 23.4, closed in the supremum norm. Moreover, it is equicontinuous by the continuity of the variation of x. The Arzelà-Ascoli theorem now implies that $\mathcal{K}_T$ is compact in the topology of the uniform convergence on J_T.

We now claim that F is continuous on $\mathcal{K}_T$ with respect to the uniform convergence. Indeed, if $y \in \mathcal{K}_T$ and $\{y^{(n)}\}_{n\in\mathbb{N}} \in \mathcal{K}_T^{\mathbb{N}}$ are such that $\|y^{(n)} - y\|_{\infty,J_T} \to 0$, then the continuity of h on $\{z \in \mathbb{R}\colon |z - y_0| \leqslant r\}$ implies $\|h \circ y^{(n)} - h \circ y\|_{\infty,J_T} \to 0$. But that gives

$$\|F(y^{(n)}) - F(y)\|_{\infty,J_T} \leqslant \|h \circ y^{(n)} - h \circ y\|_{\infty,J_T} V^1(x, J_T) \underset{n\to\infty}{\longrightarrow} 0. \tag{24.7}$$

So F is a continuous map of a compact subset of a Banach space into itself. By Schauder's fixed point theorem, F admits a fixed point; i.e., $\exists y \in \mathcal{K}_T\colon F(y) = y$. □

Remark 24.2 If x and y are general valued, the reliance on Arzelà-Ascoli's theorem requires that the space $\mathscr{Y}$ be finitely dimensional or that $h(y)$ is a compact operator.

24.2 Generalized Peano theorem.

Let us now think of how to generalize the above proof with the help of the Young integral. First, by Lemma 23.8 we cannot expect the function y to be more regular than x, i.e., y is at best of finite p-variation if x is of finite p-variation. For the Young integral to exist we then need that $h \circ y$ is of finite q-variation for some $q \geqslant 1$ with $1/q + 1/p > 1$. This shows that assuming just continuity for h is then not sufficient; we need more regularity than that. For this we observe:

Lemma 24.3 *Let $f \in \mathcal{V}^p(I)$ and let $h\colon \mathbb{R} \to \mathbb{R}$ be α-Hölder on $\mathrm{Ran}(f)$; explicitly, there exists $K \geqslant 0$ and $\alpha \in (0, 1]$ such that*

$$\forall x, y \in \mathrm{Ran}(f)\colon |h(x) - h(y)| \leqslant K|x - y|^\alpha \tag{24.8}$$

Then $h \circ f \in \mathcal{V}^{p/\alpha}(I)$ and

$$\|h \circ f\|_{p/\alpha,I} \leqslant K\|f\|_{p,I}^\alpha \tag{24.9}$$

Proof. Let $s, t \in I$. Then

$$\big|h \circ f(t) - h \circ f(s)\big|^{p/\alpha} \leqslant K^{p/\alpha}\big|f(t) - f(s)\big|^p \tag{24.10}$$

and so

$$V^{p/\alpha}(h \circ f, I, \Pi) \leqslant K^{p/\alpha} V^{p\alpha}(f, I, \Pi) \tag{24.11}$$

holds for any partition Π of I. This shows $V^{p/\alpha}(f, I) \leqslant K^{p/\alpha} V^p(f, I)$. The claim follows from the definition of the norms. □

An α-Hölder continuous h thus takes function y of finite p-variation and turns it into a function of finite p/α-variation. We thus seem to need that x is of finite p-variation for some $p \geqslant 1$ with $1/p + 1/(p/\alpha) = (1 + \alpha)/p > 1$, i.e., $p < 1 + \alpha$. This is exactly the regime in which we will prove the result:

Theorem 24.4 (Generalized Peano's theorem) *Let $h\colon \mathbb{R} \to \mathbb{R}$ be locally α-Hölder continuous for some $\alpha \in (0,1]$ and let $y_0 \in \mathbb{R}$ and $x \in \mathcal{V}^p([0,t])$ for all $t > 0$ with some $p < 1+\alpha$. Then there exists $T > 0$ and a function $y \in V^p([0,T])$ such that*

$$\forall t \in [0,T]\colon \quad y(t) = y_0 + \int_0^t h \circ y \,\mathrm{d}x, \tag{24.12}$$

where the integral exists thanks to $h \circ y \in V^{p/\alpha}([0,T])$ and thanks to $\frac{1}{p} + \frac{\alpha}{p'} > 1$.

We start with:

Lemma 24.5 *Let $p \geqslant 1$ and suppose $\mathcal{K} \subseteq \mathcal{V}^p(I)$ is bounded in $\|\cdot\|_{\mathcal{V}^p(I)}$-norm and uniformly equicontinuous. Then*

$$\forall q > p\colon \ \mathcal{K} \text{ is precompact in } \mathcal{V}^q(I) \tag{24.13}$$

Proof. Let $\{y^{(n)}\}_{n\in\mathbb{N}} \in \mathcal{K}^{\mathbb{N}}$. The boundedness in $\|\cdot\|_{\mathcal{V}^p(I)}$-norm implies boundedness in the supremum norm. As $\mathcal{K}$ is assumed uniformly equcontinuous, the Arzelà-Ascoli theorem gives existence of a strictly increasing sequence $\{n_k\}_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that $y^{(n_k)}$ converges in supremum norm. But the oscillation bound from Lemma 23.2(4) then gives

$$\|y^{(n_k)} - y^{(n_\ell)}\|_{\mathcal{V}^q(I)} \leqslant \|y^{(n_k)} - y^{(n_\ell)}\|_{\infty,I}^{1-p/q}\Bigl(1 + \|y^{(n_k)} - y^{(n_\ell)}\|_{p,I}^{q/p}\Bigr) \underset{k,\ell\to\infty}{\longrightarrow} 0 \tag{24.14}$$

proving that $\{y^{(n_k)}\}_{k\in\mathbb{N}}$ is Cauchy in $\mathcal{V}^q(I)$. Hence $\mathcal{K}$ has compact closure in $\mathcal{V}^q(I)$. □

Proof of Theorem 24.4. We will follow the blueprint of the previous proof. Fix any $r > 0$ and abbreviate $B(y_0,r) := \{z \in \mathbb{R}\colon |z-y_0| \leqslant r\}$. Let K be such that

$$\forall z,\tilde z \in B(y_0,r)\colon \ \bigl|h(z)\bigr| \leqslant \ \wedge \bigl|h(z)-h(\tilde z)\bigr| \leqslant K|z-\tilde z|^\alpha \tag{24.15}$$

Now pick $T > 0$ be such that

$$\bigl(K + C_{p,p/\alpha}Kr^\alpha\bigr)\|x\|_{p,J_T} \leqslant r \tag{24.16}$$

where, as before, $J_T := [0,T]$. By Theorem 22.2, the fact that $\frac{1}{p} + \frac{\alpha}{p} > 1$ implies that F in (24.3) is well defined for all $y \in \mathcal{V}^p(J)$.

Assuming $\|y-y_0\|_{\infty,J_T} \leqslant r$ and $\|y\|_{p,J_T} \leqslant r$, using (24.16) we now get

$$\|F(y)-y_0\|_{\infty,J_T} \leqslant \bigl|h(y_0)\bigr|\|x\|_{p,J_T} + C_{p,p/\alpha}K\|x\|_{p,J}\|y\|_{p,J_T}^\alpha \leqslant r \tag{24.17}$$

and, similarly, for all intervals $I \subseteq J_T$,

$$\|F(y)\|_{p,I} \leqslant \|h\circ y\|_{\infty,I}\|x\|_{p,I} + C_{p,p/\alpha}K\|x\|_{p,I}\|y\|_{p,J}^\alpha \leqslant \frac{r}{\|x\|_{p,J_T}}\|x\|_{p,I} \tag{24.18}$$

The function F thus maps

$$\mathcal{K}_T := \left\{ y \in V^p(J)\colon \begin{array}{l} y(0) = y_0,\ \|y-y_0\|_{\infty,J_T} \leqslant r \\ \|y\|_{p,J_T} \leqslant r, \\ \forall I \subseteq J\colon \ \|y\|_{p,I} \leqslant \dfrac{r}{\|x\|_{p,J_T}}\|x\|_{p,I} \end{array} \right\} \tag{24.19}$$

into itself.

Now pick $p' \in (p, 1+\alpha)$ and note that, by Lemma 24.5, $\mathcal{K}_T$ is precompact in $\mathcal{V}^{p'}(J_T)$. The lower-semicontinuity of the p-variation norms under pointwise convergence then shows that, in fact, $\mathcal{K}_T$ is closed and thus compact in $\mathcal{V}^{p'}(I)$. We thus have to show that F is continous on $\mathcal{K}_T$ with respect to $\|\cdot\|_{p',J_T}$.

Let $\{y^{(n)}\}_{n\in\mathbb{N}} \in \mathcal{K}_T^{\mathbb{N}}$ and $y \in \mathcal{K}_T$ be such that $\|y^{(n)} - y\|_{p',J_T} \to 0$. The fact that $y^{(n)}(0) = y_0 = y(0)$ then gives uniform convergence $\|y^{(n)} - y\|_{\infty,J_T} \to 0$. In light of the Hölder continuity of h, that also implies the uniform convergence under composition with h, i.e., $\|h\circ y^{(n)} - h\circ y\|_{\infty,J_T} \to 0$. By the fact that

$$\begin{aligned} \|h\circ y^{(n)} - h\circ y\|_{p/\alpha,J_T} &\leqslant \|h\circ y^{(n)}\|_{p/\alpha,J_T} + \|h\circ y\|_{p/\alpha,J_T} \\ &\leqslant K\big(\|y^{(n)}\|_{p,J_T}^{\alpha} + \|h\circ y\|_{p,J_T}^{\alpha}\big) \leqslant 2Kr^{\alpha} \end{aligned} \tag{24.20}$$

the oscillation bound implies

$$\|h\circ y^{(n)} - h\circ y\|_{p'/\alpha,J_T} \leqslant \|h\circ y^{(n)} - h\circ y\|_{\infty,J_T}^{1-p/p'} (2Kr^{\alpha})^{p'/p} \underset{n\to\infty}{\longrightarrow} 0 \tag{24.21}$$

As $\frac{1}{p} + \frac{\alpha}{p'} > 1$, we get $\|F(y^{(n)}) - F(y)\|_{p,I} \to 0$ and thus also $\|F(y^{(n)}) - F(y)\|_{p',I} \to 0$. Since F is a continuous function on a compact subset of a Banach space, the Schauder fixed point theorem implies existence of $y \in \mathcal{K}_T$ such that $F(y) = y$. □

Note that the above theorems prove existence of a local solution; one has to patch such local solutions together to obtain a maximal (i.e., not further extendable) solution. A more significant drawback of Peano's approach is that there is no uniqueness.