23. Functions of finite p-variation

As an application of the Young integral, here we develop tools for proving existence of solutions to differential equation driven by moderately rough force terms.

23.1 Spaces of finite p-variation.

Young's criterion of Stieltjes integrability opens up new avenues to prove existence of solutions to ordinary differential equations of the form

$$\mathrm{d}y_t = h(t, y_t)\mathrm{d}x_t + g(t, x_t)\mathrm{d}t. \tag{23.1}$$

which we take as a shorthand for the corresponding integral form

$$y_t = y_0 + \int_0^t h(s, y_s)\mathrm{d}x_s + \int_0^t g(s, x_s)\mathrm{d}s \tag{23.2}$$

for y_0 denoting the initial value. The function x should be understood as a driving force that, in general, has no reasons to be smooth. Note that both x and y are generally normed vector space-valued, with the integral interpreted as in Remark 22.9. By standard tricks, this allows us to include also higher order differential equations and, quite conveniently, absorb the second integral into the first.

It turns out that the regularity of the Stieltjes integral is, whenever it exists, determined by the regularity of the driving term. Specifically, for driving force process x such that $V^p(x, [0, t]) < \infty$, the best we can expect that $V^p(y, [0, t]) < \infty$. Assuming h to be Lipschitz, this translates into $V^p(h(\cdot, y), [0, t]) < \infty$ which means that the integral will make sense as soon as $\frac{1}{p} + \frac{1}{p} > 1$, i.e., $p < 2$. The question is whether this is also the condition under which a solution to above ODE exists. In order to answer that we need to delve deeper into the theory of spaces of finite p-variation.

We start with some definitions: Given a closed interval $I \subseteq \mathbb{R}$, let $C(I)$ denote the space of continuous functions $f\colon I \to \mathbb{R}$ and let $C_0(I)$ be the set of functions that vanish at the left endpoint of I. Given $p > 0$, set

$$\mathcal{V}^p(I) := \bigl\{f \in C(I)\colon V^p(f, I) < \infty\bigr\} \tag{23.3}$$

and let

$$\mathcal{V}^p_0(I) := \bigl\{f \in C_0(I)\colon V^p(f, I) < \infty\bigr\} \tag{23.4}$$

Clearly, these are linear vector space with respect to pointwise addition and scalar multiplication. For a function $f\colon I \to \mathbb{R}$, denote

$$\|f\|_{p,I} := V^p(f, I)^{1/p} \tag{23.5}$$

and

$$\|f\|_{\mathcal{V}^p(I)} := \sup_{t\in I}\bigl|f(t)\bigr| + V^p(f, I)^{1/p} \tag{23.6}$$

We now claim:

Lemma 23.1 *Let $p \geqslant 1$. Then the map $f \mapsto \|f\|_{p,I}$ is non-negative, homogeneous and subadditive. Moreover, $\|f\|_{p,I} = 0$ is equivalent to f being constant on I. In particular, $\|\cdot\|_{p,I}$ is a norm on $\mathcal{V}^p_0(I)$ and $\|\cdot\|_{\mathcal{V}^p(I)}$ is a norm on $\mathcal{V}^p(I)$.*

Proof. The homogeneity and non-negativity is immediate from the definition. For subadditivity we note that, given any partition Π of I, the Minkowski inequality gives

$$V^p(f+g,I,\Pi)^{1/p} \leqslant V^p(f,I,\Pi)^{1/p} + V^p(g,I,\Pi)^{1/p} \tag{23.7}$$

Bounding the right-hand side by the suprema with respect to Π yields $\|f+g\|_{p,I} \leqslant \|f\|_{p,I} + \|g\|_{p,I}$. The approximation using finite partitions also shows that $\|f\|_{p,I} = 0$ implies that f is constant.

The first term on the right of (23.6) is the usual supremum norm which is also non-negative, homogeneous and subadditive. The sum of the two vanishes only if the function vanishes and so $\|\cdot\|_{\mathcal{V}^p(I)}$ is a norm on $\mathcal{V}^p(I)$. Noting that a constant function that vanishes at an endpoint of I vanishes everywhere, $\|\cdot\|_{p,I}$ itself is a norm on $\mathcal{V}^p_0(I)$. $\square$

Next we observe the following properties of the norm $\|\cdot\|_{p,I}$:

Lemma 23.2 *Let $I \subseteq \mathbb{R}$ be a closed bounded interval and let $f\colon I \to \mathbb{R}$ be a function. Then*

(1) $\forall 1 \leqslant p < q\colon\ \|f\|_{p,I} \geqslant \|f\|_{q,I}$
(2) $p \mapsto \log V^p(f,I)$ *is convex and continuous on the interior of* $\{p \geqslant 1\colon V^p(f,I) < \infty\}$
(3) $\forall p \geqslant 1\colon\ \operatorname{osc}(f,I) \leqslant \|f\|_{p,I}$
(4) $\forall 1 \leqslant p < q\colon\ \|f\|_{q,I} \leqslant \operatorname{osc}(f,I)^{1-p/q}\|f\|_{p,I}^{q/p}$

Proof. To get (1), recall that $\sum_{i=1}^n x_i \leqslant (\sum_{i=1}^n x_i^\alpha)^{1/\alpha}$ whenever $n \geqslant 1$, $x_1,\dots,x_n \geqslant 0$ and $\alpha \in (0,1]$. Then observe that, for any $p < q$ and any partition Π of I, this gives $V^q(f,I,\Pi) \leqslant V^p(f,I,\Pi)^{q/p}$. Taking suprema then shows $\|f\|_{q,I} \leqslant \|f\|_{q,I}$.

For (2) we similarly observe that $p \mapsto \log V^p(f,I,\Pi)$ is finite whenever f is not constant on the partition points and convex by Hölder's inequality. As the supremum of any collection of continuous convex functions is continuous and upper-semicontinuous on the set it is finite, so is $p \mapsto \log V^p(f,I)$. Now recall that a convex function is necessarily continuous on the interior of its domain.

The bounds (3) and (4) follow from the corresponding bounds for $V^q(f,I,\Pi)$ and $V^p(f,I,\Pi)$ whose details we leave to the reader. $\square$

Note that (3) above can be supplemented by the limit statement

$$\operatorname{osc}(f,I) = \lim_{p\to\infty} \|f\|_{p,I} \quad \text{whenever r.h.s. finite} \tag{23.8}$$

which permits us to think of $C_0(I)$ as the space $\mathcal{V}^\infty_0(I)$ with the corresponding norm played by $f \mapsto \operatorname{osc}(f,I)$ which is equivalent to the supremum norm. As a consequence of the above lemma, we get:

Corollary 23.3 *For all bounded closed intervals $I \subseteq \mathbb{R}$ and all $1 \leqslant p < q < \infty$,*

$$\mathrm{BV}(I) = \mathcal{V}^1_0(I) \subseteq \mathcal{V}^p_0(I) \subseteq \mathcal{V}^q_0(I) \subseteq C_0(I) \tag{23.9}$$

with the embeddings continuous with respect to the corresponding norms.

We refer to $\mathcal{V}^p(I)$ and $\mathcal{V}^p_0(I)$ as the *spaces of finite p-variation*. In order to understand their metric structure better, we note:

Lemma 23.4 (Lower semicontinuity of $\|\cdot\|_{p,I}$) *Let f and $\{f_n\}_{n\in\mathbb{N}}$ be functions $I\to\mathbb{R}$ such that $f_n\to f$ pointwise. Then*

$$\|f\|_{p,I}\leqslant\liminf_{n\to\infty}\|f_n\|_{p,I} \tag{23.10}$$

Proof. Suppose first that $\|f\|_{p,I}$. Given $\epsilon>0$ there is a partition Π such that $V^p(f,I,\Pi)\geqslant\|f\|_{p,I}^p-\epsilon$. The pointwise convergence then gives

$$\liminf_{n\to\infty}\|f_n\|_{p,I}^p\geqslant\liminf_{n\to\infty}V^p(f_n,I,\Pi)=V^p(f,I,\Pi)\geqslant\|f\|_{p,I}^p-\epsilon \tag{23.11}$$

proving the statement in this case by taking $\epsilon\downarrow 0$. If instead $\|f\|_{p,I}=\infty$, then given any $M>0$ there exists a partition Π such that $V^p(f,I,\Pi)\geqslant M$. The same inequality then shows $\liminf_{n\to\infty}\|f_n\|_{p,I}\geqslant M$ proving the claim by taking $M\to\infty$. $\square$

Hereby we get:

Lemma 23.5 *For each bounded closed interval $I\subseteq\mathbb{R}$ and each $p\geqslant 1$, the normed spaces $(\mathcal{V}^p(I),\|\cdot\|_{\mathcal{V}^p(I)})$ and $(\mathcal{V}_0^p(I),\|\cdot\|_{p,I})$ are Banach spaces.*

Proof. Fix $p\geqslant 1$ and let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence from $\mathcal{V}_0^p(I)$. This means that $\sup_{m,n\geqslant N}\|f_n-f_m\|_{p,I}\to 0$ as $N\to\infty$ which by the oscillation bound in Lemma 23.2(3) implies existence of $f(t):=\lim_{n\to\infty}f_n(t)$ that, by Lemma 23.4 and $\sup_{n\in\mathbb{N}}\|f_n\|_{p,I}<\infty$, obeys $f\in\mathcal{V}_0^p(I)$. Invoking Lemma 23.4 one more time, the fact that $f_n-f_m\to f_n-f$ pointwise as $m\to\infty$ shows

$$\|f_n-f\|_{p,I}\leqslant\liminf_{m\to\infty}\|f_n-f_m\|_{p,I}\underset{n\to\infty}{\longrightarrow}0, \tag{23.12}$$

where the limit on the right follows from the Cauchy property of $\{f_n\}_{n\in\mathbb{N}}$. It follows that $(\mathcal{V}_0^p(I),\|\cdot\|_{p,I})$ is complete and so is a Banach space. The proof for $(\mathcal{V}^p(I),\|\cdot\|_{\mathcal{V}^p(I)})$ is analogous except that the presence of the supremum norm gives pointwise convergence $f_n\to f$ without the need to call on the oscillation bound in Lemma 23.2(3). $\square$

23.2 Examples.

The reader may wonder at this point whether simple examples of functions exist that are members of $\mathcal{V}^p(I)$ for some $p>1$ but not of $\mathcal{V}^q(I)$ for any $q<p$. A simple criterion for containment in $\mathcal{V}^p(I)$ is given in:

Lemma 23.6 *Suppose that $f\colon I\to\mathbb{R}$ is α-Hölder continuous for some $\alpha\in(0,1]$ in the sense that $\exists K\geqslant 0\,\forall t,s\in I\colon |f(t)-f(s)|\leqslant K|t-s|^\alpha$. Then $f\in\mathcal{V}^{1/\alpha}(I)$ with $\|f\|_{1/\alpha,I}\leqslant K|I|^{1/\alpha}$.*

Proof. For $p:=1/\alpha$ the Hölder continuity translates to $|f(t)-f(s)|^p\leqslant K^p|t-s|$ which shows that $V^p(f,I,\Pi)\leqslant K^p|I|$ for any partition Π of I. Hence $\|f\|_{1/\alpha,I}\leqslant K|I|^{1/\alpha}$. $\square$

A deterministic example of a function with non-trivial p-variation is the *Weierstrass function* that we write in the form

$$W_\alpha(t):=\sum_{n\geqslant 0}b^{-\alpha n}\cos(2\pi b^n t) \tag{23.13}$$

where $b > 1$ and $\alpha > 0$. As it turns out that, for $\alpha \in (0,1)$, the function W_α is uniformly α-Hölder continuous. To see this, note that $|\cos(t) - \cos(s)| \leqslant \min\{2, |t-s|\}$. Denoting $n_0 = n_0(t,s) := \lfloor \log_b(1/|t-s|) \rfloor$, a calculation shows

$$\begin{aligned}\bigl|W_\alpha(t) - W_\alpha(s)\bigr| &\leqslant \sum_{n=0}^{n_0} b^{(1-\alpha)n}|t-s| + 2\sum_{n>n_0} b^{-n\alpha} \\ &\leqslant \frac{1}{1-b^{-(1-\alpha)}} b^{(1-\alpha)n_0}|t-s| + \frac{2}{1-b^{-\alpha}} b^{-\alpha(n_0+1)} \leqslant K|t-s|^\alpha\end{aligned} \tag{23.14}$$

for $K := 3(1-b^{-\min\{\alpha,1-\alpha\}})^{-1}$. It follows that $W_\alpha \in V^{1/\alpha}(I)$ for any bounded closed interval $I \subseteq \mathbb{R}$. (The situation for $\alpha = 1$ is already more complicated.)

To find the optimal p for which $W_\alpha \in \mathcal{V}^p(I)$ we note the following partial reversal of the above criterion:

Lemma 23.7 *Suppose that $f\colon I \to \mathbb{R}$, $\alpha \in (0,1]$, and $c > 0$ are such that the set $\mathcal{J}$ of non-degenrate intervals $[s,t] \subseteq I$ for which there exists $[s',t'] \subseteq [s,t]$ with $|f(t') - f(s')| \geqslant c|t-s|^\alpha$ has the Cousin property: namely, for each $x \in I$ there exists $\delta > 0$ such that all subintervals of I of length at most δ containing x belong to $\mathcal{J}$. Then $f \notin \mathcal{V}^p(I)$ for $p < 1/\alpha$.*

Proof. As is well known from the theory of Henstock-Kurzweil integral, the Cousin property ensures existence of a sequence of partitions $\{\Pi_n\}_{n\in\mathbb{N}}$ of I with $\|\Pi_n\| \to 0$ such that the intervals of Π_n belong to $\mathcal{J}$. Writing $\Pi_n = \{t_i\}_{i=0}^{m_n}$, for each $i = 1, \dots, m_n$ let $[s_i', t_i'] \subseteq [t_{i-1}, t_i]$ be such that $|f(t_i') - f(s_i')| \geqslant c|t_i - t_{i-1}|^\alpha$. Adding the points $s_1', t_1', \dots, s_n', t_n'$ to Π_n defines a partition Π_n' such that

$$V^p(f, I, \Pi_n') \geqslant c^p \sum_{i=1}^{n} |t_i - t_{i-1}|^{p\alpha} \geqslant c^p \|\Pi_n\|^{p\alpha-1}|I|, \tag{23.15}$$

where we used that $|t_i - t_{i-1}|^{p\alpha-1} \geqslant \|\Pi_n\|^{p\alpha-1}$ thanks to $p\alpha < 1$. The right-hand side diverges as $n \to \infty$. □

Building on the argument from (23.14), the difference $W_\alpha(t) - W_\alpha(s)$ equals

$$b^{-\alpha n_0} 2\sin\bigl(\tfrac{1}{2}b^{n_0}(t-s)\bigr)\sin\bigl(\tfrac{1}{2}b^{n_0}(t+s)\bigr) \tag{23.16}$$

plus a quantity bounded by $Kb^{-\min\{\alpha,1-\alpha\}}|t-s|^\alpha$. Varying t, s through $[x-\delta, x+\delta]$ subject to $|t-s| = \delta/\pi$ then shows

$$\inf_{x\in\mathbb{R}} \liminf_{\delta\downarrow 0} \frac{\operatorname{osc}(W_\alpha, [x-\delta, x+\delta])}{\delta^\alpha} > 0 \tag{23.17}$$

once $b \geqslant b_0$ for some constant b_0 depending on α. This now readily implies that W_α satisfies the premise of Lemma 23.7 which is enough to conclude that $W_\alpha \notin \mathcal{V}^p([0,2\pi))$ for $p < 1/\alpha$. We leave the details to the reader.

The above deterministic example is nice because it allows for explicit calculations, but as probabilists, we are interested in random examples as well. The standard Brownian motion $\{B_t\colon t \geqslant 0\}$ is a simplest example because, as discussed earlier in this course, it is not of of finite p-variation on any interval for $p > 2$ but not for $p \leqslant 2$. This can in fact be iterated to generate more singular examples: Let $\{B^n_t\colon t \geqslant 0\}$ be samples of independent two-sided standard Brownian motions and define $\{W^{(n)}_t\colon t \geqslant 0\}$ recursively by

$$W^{(1)}_t := B^{(1)}_t \;\wedge\; \forall n \geqslant 1\colon\; W^{(n+1)}_t := W^{(n)}_{B^{(n+1)}_t} \tag{23.18}$$

Then, by a similar reasoning as for the Weierstrass function above, we have $W^{(n)} \in \mathcal{V}^p(I)$ for $p > 2^n$ but not for $p \leqslant 2^n$.

Perhaps more natural is the following generalization of the standard Brownian motion called the *Fractional Brownian Motion*: Consider a stochastic process $\{X_t\colon t \geqslant 0\}$ that is multivariate Gaussian with mean zero and covariance given by

$$\operatorname{Cov}(X_t, X_s) = \frac{1}{2}\Big(s^{2H} + t^{2H} - |t-s|^{2H}\Big) \tag{23.19}$$

where $H \in (0,1)$ is a parameter called the *Hurst index*. (The function on the right is positive definite and so the process exists by the Kolmogorov Extension Theorem. For $H = 1/2$ the right-hand side equals $\min\{s,t\}$ and so X is a standard Brownian motion in this case.) A calculation shows that

$$\operatorname{Var}(X_t - X_s) = |t-s|^{2H} \tag{23.20}$$

and so the process has stationary increments. The same argument as for the standard Brownian motion shows that X admits a version that is α-Hölder continuous for each $\alpha < H$, and so the process is of finite p-variation for all $p > 1/H$. A more subtle argument then shows that it is not of finite p-variation for $p \leqslant 1/H$.

Examples of functions of finite p-variation naturally arise from the Young integral involving such functions. This is shown in:

Lemma 23.8 *Let $f \in \mathcal{V}^p([a,b])$ and $g \in \mathcal{V}^q([a,b])$ for $p,q \geqslant 1$ with $1/p + 1/q > 1$. Define $h(t) := \int_a^t g\mathrm{d}f$ for $t \in [a,b]$. Then $h \in \mathcal{V}^p(I)$ and*

$$\|h\|_{p,I} \leqslant C_{p,q}\, \|g\|_{\mathcal{V}^q(I)}\, \|f\|_{p,I} \tag{23.21}$$

holds with $C_{p,q} = 1 + \zeta(1/p + 1/q)$.

Proof. It is well known that the Stieltjes integral of g with respect to f is continuous in the limits whenever f is continuous. The inequality in Corollary 22.7 shows that, for all $s,t \in I$ with $s \leqslant t$,

$$\begin{aligned} \big|h(t) - h(s)\big| &\leqslant \sup_{u\in[s,t]} \big|g(u)\big|\,\big|f(t) - f(s)\big| + C_{p,q} V^q\big(g,[s,t]\big)^{1/q}\, V^p\big(f,[s,t]\big)^{1/p} \\ &\leqslant C_{p,q}\|g\|_{\mathcal{V}^q(I)}\, V^p\big(f,[s,t]\big)^{1/p} \end{aligned} \tag{23.22}$$

where we used that $C_{p,q} \geqslant 1$. For any partition $\Pi = \{t_i\}_{i=0}^n$ of I, we thus get

$$V^p(h, I, \Pi) \leqslant \big(C_{p,q}\|g\|_{\mathcal{V}^q(I)}\big)^p \sum_{i=1}^n V^p\big(f,[t_{i-1},t_i]\big) \tag{23.23}$$

The sum on the right is bounded by $V^p(f,I)$ which then gives the claim. $\square$

23.3 Invariance under reparametrization.

Unlike Hölder continuity, the concept of p-variation has the following property:

Lemma 23.9 (Invariance under reparametrization) *Let $\varphi\colon I \to I$ be continuous, strictly increasing and bijective. Then for all $p \geqslant 1$ and all $f\colon I \to \mathbb{R}$,*

$$V^p(f \circ \varphi, I) = V^p(f, I) \tag{23.24}$$

In particular, $f \in \mathcal{V}^p(I)$ is equivalent to $f \circ \varphi \in \mathcal{V}^p(I)$ and $\|f \circ \varphi\|_{p,I} = \|f\|_{p,I}$.

Proof. The properties of φ ensure that φ maps the endpoints of I onto themselves. Let $\Pi = \{t_i\}_{i=0}^n$ be a partition of I. Then $\Pi' := \{\varphi(t_i)\}_{i=0}^n$ is a partition as well and $V^p(f \circ \varphi, I, \Pi) = V^p(f, I, \Pi')$. This readily implies $V^p(f, \circ\varphi, I) \geqslant V^p(f, I)$. The opposite inequality follows from the fact that φ is invertible. □

The above allows us to make the link with Hölder continuity even more tight. First observe that the p-variation of a continuous function is continuous:

Lemma 23.10 *Let $a < b$ and $p \geqslant 1$ be reals. Given $f\colon [a,b] \to \mathbb{R}$ such that $V^p(f, [a,b]) < \infty$, let $v_f\colon [a,b] \to \mathbb{R}_+$ be defined by $v_f(t) := V^p([a,t])$ for $t > a$ and $v_f(a) := 0$. Then v_f obeys*

$$\forall a \leqslant s < t \leqslant b\colon\ \ V^p\big(f, [s,t]\big) \leqslant v_f(t) - v_f(s) \tag{23.25}$$

and, in particular, v_f is non-decreasing. Moreover,

$$f \in C[a,b] \ \Leftrightarrow\ v_f \in C[a,b] \tag{23.26}$$

We leave the proof of this to homework. The quantity v_f can actually be used to control the regularity of f. Indeed, the definition implies that, for all $t, s \in I$ with $s \leqslant t$,

$$v_f(t) - v_f(s) \geqslant V^p\big(f, [s,t]\big) \geqslant \big|f(t) - f(s)\big|^p \tag{23.27}$$

Using that we now get:

Lemma 23.11 *Let $p \geqslant 1$ and $f \in \mathcal{V}^p(I)$ be such that f is not constant on any non-degenerate closed subinterval of I. Then there exists a continuous bijection $\varphi\colon I \to I$ such that $f \circ \varphi$ is $1/p$-Hölder continuous. In fact,*

$$\forall t, s \in I\colon\ \ \big|f \circ \varphi(t) - f \circ \varphi(s)\big| \leqslant \Big(\frac{V^p(f, I)}{|I|}\Big)^{1/p} |t - s|^{1/p} \tag{23.28}$$

Proof. Write $I = [a,b]$ and let $\varphi\colon [a,b] \to [a,b]$ be the inverse of the continuous, strictly increasing bijection $t \mapsto a + \frac{b-a}{V^p(f,[a,b])} v_f(t)$ of $[a,b]$ onto itself. Then for all $a \leqslant s < t \leqslant b$,

$$\big|f \circ \varphi(t) - f \circ \varphi(s)\big|^p \leqslant v_f \circ \varphi(t) - v_f \circ \varphi(s) = \frac{V^p(f, [a,b])}{b-a}(t - s) \tag{23.29}$$

which now implies the claim. □

Note that the resulting constant in the Hölder estimate (23.28) depends only on the norm $\|f\|_{p,I}$. In particular, the constant is uniform over bounded subsets of $\mathcal{V}^p(I)$.

The invariance under reparametrization has one negative consequence: Unlike $C(I)$, the spaces $\mathcal{V}^p(I)$ for $p \in [1, \infty)$ are not separable. This is easiest to see for $p = 1$ where for each finite Borel measure μ on $[a,b]$, the function $f_\mu(t) := \mu([a,t])$ belongs to $\mathcal{V}^1([a,b])$. Given two such measures μ and ν, we have $\|f_\mu - f_\nu\|_{1,[a,b]} = |\mu - \nu|([a,b])$ which, if μ and ν are mutually singular, equals $\mu([a,b]) + \nu([a,b])$. Now we just need to find an uncountable family of mutually singular Borel probability supported in $[a,b]$, a task that we will leave to the reader.