22. Young integral

We will now finally ready to move to the most recent developments in stochastic analysis. This will require introduction of T. Lyons' theory of rough paths from 1998 which in turn extends the theory of so called Young integral, developed by L.C. Young in the 1930s, that we will start with here. Both theories are concerned with integrals of the form $\int_a^b f \mathrm{d}g$ whose treatment dates back to T.J. Stieltjes in 1890s.

22.1 Stieltjes integral.

The Young integral is actually a code word for a *criterion* for Stieltjes integrability. We recall the following concepts: Given two reals $a < b$, a marked partition is the pair $\Pi = (\{t_i\}_{i=0}^n, \{t_i^\star\}_{i=1}^n\})$ such that

$$t_0 = a < t_1 < \cdots < t_n = b \tag{22.1}$$

and

$$\forall i = 1, \dots, n\colon\ \ t_i^\star \in [t_{i-1}, t_i] \tag{22.2}$$

Given a marked partition Π as above and functions $f, g\colon [a,b] \to \mathbb{R}$,

$$S(f,g,\Pi) := \sum_{i=1}^n f(t_i^\star)\bigl[g(t_i) - g(t_{i-1})\bigr] \tag{22.3}$$

then denotes the associated Riemann-Stieltjes sum. We will write

$$\|\Pi\| := \max_{i=1,\dots,n} |t_i - t_{i-1}| \tag{22.4}$$

for the mesh of the partition Π. Now recall:

Definition 22.1 (Stieltjes integrability) *We say that f is Stietljes integrable with respect to g on $[a,b]$ if there exists $L \in \mathbb{R}$ and, for any $\epsilon > 0$, there exists $\delta > 0$ such that for all marked partitions Π of $[a,b]$,*

$$\|\Pi\| < \delta \ \Rightarrow\ \bigl|S(f,g,\Pi) - L\bigr| < \epsilon \tag{22.5}$$

The (necessarily unique) L with this property is then denoted $\int_a^b f \mathrm{d}g$ and called the Riemann-Stieltjes integral of f with respect to g on $[a,b]$.

A standard criterion for Stieltjes integrability is that f is continuous and g is finite first variation (a.k.a. bounded variation). However, the concept as stated above is symmetric: f is Riemann-Stieltjes integrable with respect to g if and only if g is Riemann-Stieltjes integrable with respect to f. Hence, we get integrability also for g continuous and f of bounded variation.

This "swap" suggests that one can trade increased regularity of one function for decreased regularity of the other. As discovered by L.C. Young, the correct way to expressed this trade-off is using the p-variation. For a given partition $\Pi = \{t_0 = a < t_1 < \cdots < t_n = b\}$ of $[a,b]$, this is defined by

$$V^p(f,[a,b],\Pi) := \sum_{i=1}^n \bigl|f(t_i) - f(t_{i-1})\bigr|^p \tag{22.6}$$

We then set

$$V^p(f,[a,b]) := \sup_{\Pi} V^p(f,[a,b],\Pi) \tag{22.7}$$

and say that f is of finite p-variation if $V^p(f) < \infty$. (To prevent later confusion, note that $\Pi \mapsto V^p(f,[a,b],\Pi)$ is monotone under refinements of the partition only for $p \leqslant 1$, which is the less interesting regime.) We then have:

Theorem 22.2 (L.C. Young's criterion for Stieltjes integrability) *Let $a < b$ be reals and let $f,g\colon [a,b] \to \mathbb{R}$ be functions with no common discontinuity points. Assume there exist reals $p,q \geqslant 1$ with*

$$\frac{1}{p} + \frac{1}{q} > 1 \tag{22.8}$$

such that

$$V^p(f,[a,b]) < \infty \ \wedge \ V^q(g,[a,b]) < \infty. \tag{22.9}$$

Then f is Riemann-Stieltjes integrable with respect to g on $[a,b]$.

Note that this formally interpolates between the case of f continuous ($p = \infty$) and g of bounded variation ($q = 1$) and the case of g continuous ($q = \infty$) and f bounded variation ($p = 1$). The boundary case $1/p + 1/q = 1$ is excluded because the conclusion actually fails in that case. This is easiest to see when $p = q = 2$ because then for any partition $\Pi = \{t_i\}_{i=0}^n$ of $[a,b]$ and with Π' denoting the marked partition with $t_i^\star := t_i$ and Π'' the marked partition with $t_i^\star := t_{i-1}$, we have

$$S(f,f,\Pi') - S(f,f,\Pi'') = V^2(f,[a,b],\Pi) \tag{22.10}$$

The Riemann-Stieltjes integrability of f with respect to f fails whenever $V^2(f,[a,b],\Pi)$ does not vanish as $\|\Pi\| \to 0$.

22.2 The Love-Young inequality.

Theorem 22.2 appears in L.C. Young's paper "An inequality of Hölder type, connected with Stieltjes integration" published in Acta Mathematica in 1936. A key step is the following inequality for which he credits (with a different derivation) R.E. Love who studied it "at [his] suggestion." (R.E. Love, sometimes misquoted as Loeve in this context, went on to a successful career at University of Melbourne. He seems to have no joint paper with L.C. Young despite an announcement of one in the paper above.)

Lemma 22.3 (Love-Young inequality) *Let $a < b$ be reals and $f,g\colon [a,b] \to \mathbb{R}$ functions. Let $p,q > 0$ obey $1/p + 1/q > 1$. Then for any $n \geqslant 1$, any marked partition $(\{t_i\}_{i=0}^n, \{t_i^\star\}_{i=1}^n)$ of $[a,b]$ and any $t^\star \in [a,b]$,*

$$\begin{aligned}\left| f(t^\star)\bigl[g(b) - g(a)\bigr] - \sum_{i=1}^n f(t_i^\star)\bigl[g(t_i) - g(t_{i-1})\bigr]\right| \\ \leqslant \Bigl(1 + \zeta_n\bigl(1/p + 1/q\bigr)\Bigr)\, V^p\bigl(f,[a,b]\bigr)^{1/p}\, V^q\bigl(g,[a,b]\bigr)^{1/q},\end{aligned} \tag{22.11}$$

where $\zeta_n(s) := \sum_{k=1}^{n-1} k^{-s}$.

As in L.C. Young's paper, we start by the following simple observation:

Lemma 22.4 *For all reals $p,q>0$, an integer $n\geqslant 1$ and reals $a_1,\dots,a_n,b_1,\dots,b_n\geqslant 0$,*

$$\min_{i=1,\dots,n} a_ib_i \leqslant \Bigl(\prod_{i=1}^n a_i^p\Bigr)^{\frac{1}{np}}\Bigl(\prod_{i=1}^n b_i^q\Bigr)^{\frac{1}{nq}} \leqslant \biggl(\frac1n\sum_{i=1}^n a_i^p\biggr)^{1/p}\biggl(\frac1n\sum_{i=1}^n b_i^q\biggr)^{1/q} \tag{22.12}$$

Proof. The first inequality is immediate and the second one follows by the so called AMGM inequality $(a_1\dots a_n)^{1/n}\leqslant\frac1n(a_1+\dots+a_n)$ which is readily proved from the concavity of the logarithm function. □

This now gives:

Proof of Lemma 22.3. We will prove this by induction in the number of points of the partition. The initial ($n=1$) step is simple:

$$\begin{aligned}&\Bigl|f(t^\star)\bigl[g(b)-g(a)\bigr]-f(s^\star)\bigl[g(b)-g(a)\bigr]\Bigr|\\ &\qquad\leqslant\bigl|f(t^\star)-f(s^\star)\bigr|\bigl|g(b)-g(a)\bigr|\leqslant V^p\bigl(f,[a,b]\bigr)^{1/p}\,V^q\bigl(g,[a,b]\bigr)^{1/q}\end{aligned} \tag{22.13}$$

For the induction step, let us assume that the bound holds for partitions into $n-1$ intervals and consider a marked partition $(\{t_i\}_{i=0}^n,\{t_i^\star\}_{i=1}^n)$ of $[a,b]$ into n intervals. Given any $k=1,\dots,n-1$, let $(\{s_i\}_{i=0}^{n-1},\{s_i^\star\}_{i=1}^{n-1})$ be defined by

$$(s_i,s_i^\star):=\begin{cases}(t_i,t_i^\star), & \text{if } i<k,\\ (t_{i+1},t_{i+1}^\star), & \text{if } i\geqslant k,\end{cases} \tag{22.14}$$

with $s_0:=0$. This "new" partition simply leaves out the point t_k, combines $[t_{k-1},t_k]$ and $[t_k,t_{k+1}]$ into one interval and puts $t_{k+1}^\star$ as its marked point. Consequently we get

$$\begin{aligned}&\sum_{i=1}^{n-1}f(s_i^\star)\bigl[g(s_i)-g(s_{i-1})\bigr]-\sum_{i=1}^{n}f(t_i^\star)\bigl[g(t_i)-g(t_{i-1})\bigr]\\ &\qquad=f(t_{k+1}^\star)\bigl[g(t_{k+1})-g(t_{k-1})\bigr]\\ &\qquad\qquad-f(t_k^\star)\bigl[g(t_k)-g(t_{k-1})\bigr]-f(t_{k+1}^\star)\bigl[g(t_{k+1})-g(t_k)\bigr]\\ &\qquad=\bigl[f(t_{k+1}^\star)-f(t_k^\star)\bigr]\bigl[g(t_k)-g(t_{k-1})\bigr]\end{aligned} \tag{22.15}$$

With (22.12) in sight, this now suggests to pick k as the index minimizing the absolute value of the product. The inequality (22.12) then shows

$$\begin{aligned}&\biggl|\sum_{i=1}^{n-1}f(s_i^\star)\bigl[g(s_i)-g(s_{i-1})\bigr]-\sum_{i=1}^{n}f(t_i^\star)\bigl[g(t_i)-g(t_{i-1})\bigr]\biggr|\\ &\qquad\leqslant(n-1)^{-1/p-1/q}\,V^p\bigl(f,[a,b],\Pi^\star\bigr)^{1/p}\,V^q\bigl(g,[a,b],\Pi\bigr)^{1/q}\end{aligned} \tag{22.16}$$

where we first noted that k can take only $n-1$ values and then bounded the sums of differences of f's and g's arising from (22.12) in terms of variations for the unmarked partitions $\Pi=\{t_i\}_{i=0}^n$ and $\Pi^\star:=\{t_i^\star\}_{i=0}^{n+1}$ in which $t_0^\star:=a$ and $t_{n+1}^\star:=b$.

Bounding the variations by their suprema and invoking the induction assumption that reads

$$\begin{aligned}\left|f(t^\star)\left[g(b)-g(a)\right]-\sum_{i=1}^{n-1}f(s_i^\star)\left[g(s_i)-g(s_{i-1})\right]\right|\\ \leqslant\Bigl(1+\zeta_{n-1}(1/p+1/q)\Bigr)\,V^p\bigl(f,[a,b]\bigr)^{1/p}\,V^q\bigl(g,[a,b]\bigr)^{1/q}\end{aligned}\tag{22.17}$$

we get (22.11) using the triangle inequality for the absolute value along with the fact that $\zeta_n(s)=\zeta_{n-1}(s)+(n-1)^{-s}$. □

22.3 Proof of Theorem 22.2.

We will now move to the proof of Young's criterion for Stieltjes integrability. In order to deal with an estimate in the forthcoming proof, we first note:

Lemma 22.5 *Let $f,g\colon[a,b]\to\mathbb{R}$ be such that $V^p(f,[a,b])<\infty$ and $V^q(g,[a,b])<\infty$ for some $p,q\geqslant 1$ and denote, for a partition $\Pi=\{t_0=a<t_1<\dots<t_n=b\}$ of $[a,b]$,*

$$O(f,g,\Pi):=\max_{i=1,\dots,n}\Bigl(\operatorname{osc}\bigl(f,[t_{i-1},t_i]\bigr)\operatorname{osc}\bigl(g,[t_{i-1},t_i]\bigr)\Bigr).\tag{22.18}$$

Then both f and g have only discontinuities of the first kind and, if f and g have no common discontinuities, then for each $\epsilon>0$ there is $\delta>0$ such that

$$\|\Pi\|<\delta\ \Rightarrow\ O(f,g,\Pi)<\epsilon.\tag{22.19}$$

Proof (sketch). The existence of finite variations implies that both f and g are bounded and have only discontinuities of the first kind. Assume there is a sequence $\{\Pi_n\}_{n\in\mathbb{N}}$ of partitions with mesh tending to zero yet $O(f,g,\Pi_n)\geqslant\epsilon$ for all $n\in\mathbb{N}$. Writing x_n for the left end-point in the maximizing interval, compactness of $[a,b]$ implies that $\{x_n\}_{n\in\mathbb{N}}$ has at least one accumulation point $x\in[a,b]$. But, in light of boundedness of f and g, both f and g have a discontinuity at x, contradicting our assumption. □

We also note the following fact:

Lemma 22.6 (Extended Hölder inequality) *For all $p',q'\geqslant 1$ with $1/p'+1/q'\geqslant 1$, all $n\geqslant 1$ and all $a_1,\dots,a_n,b_1,\dots,b_n\geqslant 0$,*

$$\sum_{i=1}^n a_ib_i\leqslant\biggl(\sum_{i=1}^n a_i^{p'}\biggr)^{1/p'}\biggl(\sum_{i=1}^n b_i^{q'}\biggr)^{1/q'}\tag{22.20}$$

Proof. By continuity we may assume that $p',q'>1$. Denote by $p'':=(1-1/q')^{-1}$ the Hölder conjugate of q'. Then the standard Hölder inequality gives

$$\sum_{i=1}^n a_ib_i\leqslant\biggl(\sum_{i=1}^n a_i^{p''}\biggr)^{1/p''}\biggl(\sum_{i=1}^n b_i^{q'}\biggr)^{1/q'}.\tag{22.21}$$

Denoting $\alpha:=p'/p''<1$, the concavity of $x\mapsto x^\alpha$ implies subadditivity $(x+y)^\alpha\leqslant x^\alpha+y^\alpha$ for all $x,y\geqslant 0$ and, by iteration, $(\sum_{i=1}^n x_i)^\alpha\leqslant\sum_{i=1}^n x_i^\alpha$ for all $x_1,\dots,x_n\geqslant 0$.

Using this for $x_i := a_i^{p''}$ for which $x_i^\alpha = a_i^{p'}$ now bounds the right-hand side of (22.21) by that of (22.20). (Incidentally, L.C. Young derives (22.20) directly from (22.12).) □

We are now ready to show:

Proof of Theorem 22.2. First use that $1/p + 1/q > 1$ to infer existence of $p' > p$ and $q' > q$ such that $1/p' + 1/q' > 1$ and $p'/p = q'/q$. Next let Π and Π' be marked partitions such that every interval of Π is a union of intervals from Π'. (The marked points in the two partitions can be completely unrelated.) Denoting $C := 1 + \zeta(1/p' + 1/q')$ where $\zeta(s) := \sum_{n\geqslant 1} n^{-s}$ is the Riemann zeta function, Lemma 22.3 gives

$$\big|S(f,g,\Pi) - S(f,g,\Pi')\big| \leqslant C \sum_{i=1}^n V^{p'}\big(f,[t_{i-1},t_i]\big)^{1/p'} V^{q'}\big(g,[t_{i-1},t_i]\big)^{1/q'} \tag{22.22}$$

where $t_0 = a < t_1 < \dots < t_n = b$ are the partition points of Π. Noting that

$$V^{p'}\big(f,[t_{i-1},t_i]\big) \leqslant \operatorname{osc}\big(f,[t_{i-1},t_i]\big)^{p'-p} V^{p}\big(f,[t_{i-1},t_i]\big) \tag{22.23}$$

we then use (22.20) to bound

$$\begin{aligned} &\sum_{i=1}^n V^{p'}\big(f,[t_{i-1},t_i]\big)^{1/p'} V^{q'}\big(g,[t_{i-1},t_i]\big)^{1/q'} \\ &\qquad \leqslant O(f,g,\Pi)^{1-p/p'} \left(\sum_{i=1}^n V^{p}\big(f,[t_{i-1},t_i]\big)\right)^{1/p'} \left(\sum_{i=1}^n V^{q}\big(q,[t_{i-1},t_i]\big)\right)^{1/q'} \end{aligned} \tag{22.24}$$

where we used that $1 - q/q' = 1 - p/p'$. The definition of the p-variation gives

$$\sum_{i=1}^n V^{p}\big(f,[t_{i-1},t_i]\big) \leqslant V^p\big(f,[a,b]\big) \tag{22.25}$$

with a similar bound for the second sum on the right of (22.24). Lemma 22.5 thus bounds the difference in (22.22) by a constant times $\epsilon^{1-p/p'}$ once $\|\Pi\| < \delta$.

Iterating (22.22) we get a similar observation with Π' any partition with $\|\Pi'\| < \delta$. Using a standard reasoning it follows that f is Riemann-Stieltjes integrable with respect to g on $[a,b]$, as desired. □