2. Quadratic variation

The goal of this section is to give a construction of a fundamental processes associated with continuous martingales — their quadratic variation.

2.1 Statement and uniqueness.

Examples of continuous martingales are easy to give once we have defined the standard Brownian motion $\{B_t\colon t \geqslant 0\}$. This is a continuous process with independent increments such that $B_t - B_s = \mathcal{N}(0, t-s)$ and $B_0 = 0$. Then $\{B_t\colon t \geqslant 0\}$ is a martingale but $\{B_t^2\colon t \geqslant 0\}$ is only a submartingale; i.e., a process with "upward trend." However, the process $\{B_t^2 - t\colon t \geqslant 0\}$ is again a martingale.

As it turns out, the same argument applies to a large class of continuous submartingales leading to well known *Doob-Meyer decomposition*. We will only state and prove this for submartingales that arise as squares of continuous martingales. In fact, we only need to assume local versions of these notions:

Theorem 2.1 *Let $\{M_t\colon t \geqslant 0\}$ be a continuous local martingale with respect to filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ such that $\mathcal{F}_0$ contains all P-null sets. Then there exists an adapted, continuous, non-decreasing process $\{\langle M\rangle_t\colon t \geqslant 0\}$ such that $\langle M\rangle_0 = 0$ and*

$$\bigl\{M_t^2 - \langle M\rangle_t\colon t \geqslant 0\bigr\} \text{ is a local martingale} \tag{2.1}$$

Any two processes with all these properties are indistinguishable.

Let us start with the proof of uniqueness which also enters part of the existence argument. This relies on:

Lemma 2.2 *Let $\{M_t\colon t \geqslant 0\}$ be a continuous local martingale whose a.e. sample path is of bounded variation on $[0,t]$. Then*

$$P\bigl(\forall s \leqslant t\colon M_s = M_0\bigr) = 1 \tag{2.2}$$

Proof. Given $t \geqslant 0$, $p > 0$ and a partition Π of $[0,t]$ with partition points $0 = t_0 < t_1 < \dots < t_n = t$, denote

$$V_t^{(p)}(M,\Pi) := \sum_{i=1}^n |M_{t_i} - M_{t_{i-1}}|^p \tag{2.3}$$

The first variation of M on $[0,t]$ is then given as $V_t^{(1)}(M) := \sup_\Pi V_t^{(1)}(M,\Pi)$. Since M is continuous, so is $t \mapsto V_t^{(1)}(M)$. We can thus define the stopping time

$$\tau_n := \inf\bigl\{t \geqslant 0\colon |M_t| \geqslant n \,\wedge\, V_t^{(1)}(M) \geqslant n\bigr\} \tag{2.4}$$

and observe that, by continuity and elementary facts about first variation, the stopped local martingale $\widetilde{M} := M_{\tau_n \wedge t}$ is a martingale that is bounded $|\widetilde{M}_t| \leqslant n$ and has bounded variation $V_t^{(1)}(\widetilde{M}) \leqslant n$ at all $t \geqslant 0$.

For each $s \leqslant t$ and any partition Π of $[0,s]$, the martingale property now shows

$$\begin{aligned} E\big(|\widetilde{M}_s - \widetilde{M}_0|^2\big) = E\big(V_s^{(2)}(\widetilde{M},\Pi)\big) &\leqslant E\Big(\operatorname{osc}_{\widetilde{M}}\big([0,t],\|\Pi\|\big)V_s^{(1)}(\widetilde{M},\Pi)\Big) \\ &\leqslant E\Big(\operatorname{osc}_{\widetilde{M}}\big([0,t],\|\Pi\|\big)V_s^{(1)}(\widetilde{M})\Big) \leqslant nE\Big(\operatorname{osc}_{\widetilde{M}}\big([0,t],\|\Pi\|\big)\Big) \end{aligned} \tag{2.5}$$

By continuity and boundedness of $\widetilde{M}$, the right-hand side tends to zero as $\|\Pi\| \to 0$ with the help of the Bounded Convergence Theorem. This gives $P(M_{s\wedge\tau_n} = M_0) = 1$ for each $s \in [0,1]$ and, taking countable unions,

$$\forall n \geqslant 1\colon\ P\big(\forall s \in \mathbb{Q} \cap [0, t\wedge\tau_n]\colon M_s = M_0\big) = 1. \tag{2.6}$$

The assumed continuity allows us to drop the restriction to rational $\mathbb{Q}$. Since $\tau_n \to \infty$ a.s. as $n \to \infty$ (implied by continuity and the assumption that M is of bounded variation on $[0,t]$ a.s.), we get (2.2) as desired. □

We note that continuity is essential for the statement as the example of a compensated homogeneous Poisson process (which is a discontinuous martingale with bounded variation) shows.

Proof of uniqueness in Theorem 2.1. Suppose X is a continuous process and A, A' are adapted non-decreasing continuous processes such that both $\{X_t - A_t\colon t \geqslant 0\}$ and $\{X_t - A'_t\colon t \geqslant 0\}$ are local martingales. Then also $\{A_t - A'_t\colon t \geqslant 0\}$ is a local martingale whose every path is of bounded variation on $[0,t]$, for all $t \geqslant 0$. Hence $A_t - A'_t = A_0 - A'_0$ for all $t \geqslant 0$, a.s. In particular, if also $A_0 = A'_0$, then the processes $\{A_t\colon t \geqslant 0\}$ and $\{A'_t\colon t \geqslant 0\}$ are indistinguishable. (The claim corresponds to $X_t := M_t^2$.) □

2.2 Key lemmas.

The proof of existence is based on some pretty heavy estimates that we stated in two lemmas. The starting point is the following bound:

Lemma 2.3 *Let $t \geqslant 0$ and let M be a continuous martingale such that, for some $K \geqslant 0$, we have $\sup_{t\geqslant 0}|M_t| \leqslant K$ a.s. Then for any two partitions Π and $\widetilde{\Pi}$ of $[0,t]$ with $\|\widetilde{\Pi}\| \leqslant \|\Pi\|$,*

$$E\bigg(\Big|V_t^{(2)}(M,\widetilde{\Pi}) - V_t^{(2)}(M,\Pi)\Big|^2\bigg) \leqslant 112\,K^2\left[E\Big(\operatorname{osc}_M\big([0,t],\|\Pi\|\big)^4\Big)\right]^{1/2} \tag{2.7}$$

Proof. Suppose first that $\widetilde{\Pi}$ is a subpartition of Π. Specifically, let Π have partition points $0 := t_{1,0} < t_{2,0} < \dots < t_{n+1,0} = t$ and $\widetilde{\Pi}$ has points

$$0 := t_{1,0} < \dots < t_{1,m_1} = t_{2,0} < \dots < t_{n-1,m_{n-1}} = t_{n,0} < \dots < t_{n,m_n} = t_{n+1,0} = t \tag{2.8}$$

Denoting $Z_{i,j} := M_{t_{i,j}} - M_{t_{i,j-1}}$ we get

$$V_t^{(2)}(M,\widetilde{\Pi}) = \sum_{i=1}^{n}\sum_{j=1}^{m_i} Z_{i,j}^2 \tag{2.9}$$

while

$$V_t^{(2)}(M,\Pi)=\sum_{i=1}^n\Bigl(\sum_{j=1}^{m_i}Z_{i,j}\Bigr)^2 \tag{2.10}$$

Hence

$$V_t^{(2)}(M,\Pi)-V_t^{(2)}(M,\widetilde{\Pi})=2\sum_{i=1}^n\sum_{1\leqslant j<j'\leqslant m_i}Z_{i,j}Z_{i,j'} \tag{2.11}$$

Next we want to square the expression and take expectation. This leads to the sum of expectations of the form $E(Z_{i,j}Z_{i,j'}Z_{i',k}Z_{i',k'})$, where $j<j'$ and $k<k'$. (These are meaningful because M and thus the Z's are bounded.) A key observation is that, thanks to the Z's being martingale increments, this expectation vanishes unless $k=j'$ and $i=i'$. The expectation on the left of (2.7) thus equals

$$\begin{aligned}4\sum_{i=1}^n\sum_{1<j'\leqslant m_i}\sum_{1\leqslant j,k<j'}E(Z_{i,j}Z_{i,k}Z_{i,j'}^2)&\\ =4\sum_{i=1}^n\sum_{j'=2}^{m_i}E\bigl((M_{t_{i,j'-1}}-M_{t_{i,0}})^2(M_{t_{i,j'}}-M_{t_{i,j'-1}})^2\bigr)&\end{aligned} \tag{2.12}$$

We can now bound the first difference on the right by the oscillation $\operatorname{osc}_M([0,t],\|\Pi\|)$ and resum the rest. This bounds the expression by

$$4E\Bigl(\operatorname{osc}_M\bigl([0,t],\|\Pi\|\bigr)^2V_t^{(2)}(M,\widetilde{\Pi})\Bigr). \tag{2.13}$$

We now invoke the Cauchy-Schwarz inequality which leads us to bound the second moment of $V_t^{(2)}(M,\widetilde{\Pi})$. Here we relabel the points of $\widetilde{\Pi}$ as $0=s_0<\dots<s_r=t$ as write

$$\begin{aligned}E\bigl(V_t^{(2)}(M,\Pi)^2\bigr)&=E\bigl(V_t^{(4)}(M,\Pi)^2\bigr)+2\sum_{i=1}^rE\bigl((M_{s_i}-M_{s_{i-1}})^2(M_t-M_{s_i})^2\bigr)\\ &\leqslant 4K^2E\bigl(V_t^{(2)}(M,\Pi)^2\bigr)+8K^2E\bigl(V_t^{(2)}(M,\Pi)^2\bigr)\\ &=12K^2E\bigl((M_t-M_0)^2\bigr)\leqslant 48K^4\leqslant(7K^2)^2\end{aligned} \tag{2.14}$$

Plugging this above we get the claim with 28 instead of 112 assuming $\widetilde{\Pi}$ is a refinement of Π. Considering a common refinement of the two partitions and relating each partition to that using the triangle inequality, this then implies the claim as stated. □

From this we get:

Lemma 2.4 *Assume that $\mathcal{F}_0$ contains all P-null sets. Under the assumptions of Lemma 2.3, there exists a non-decreasing continuous adapted process $\{\langle M\rangle_t\colon t\geqslant 0\}$ with $\langle M\rangle_0=0$ such that for any $t\geqslant 0$ and any sequence $\{\Pi_n\}_{n\geqslant 0}$ of partitions of $[0,t]$,*

$$\|\Pi_n\|\to 0 \quad\Rightarrow\quad V_t^2(M,\Pi_n)\ \underset{n\to\infty}{\overset{L^2}{\longrightarrow}}\ \langle M\rangle_t. \tag{2.15}$$

Moreover, $E(\langle M\rangle_t)\leqslant 4K^2$ holds for all $t\geqslant 0$.

Proof. We start with some general considerations. Let $0 \leqslant u \leqslant s \leqslant t$. The fact that M is a martingale then implies

$$E\big((M_t - M_u)^2 \,\big|\, \mathcal{F}_s\big) = (M_s - M_u)^2 + E\big((M_t - M_s)^2 \,\big|\, \mathcal{F}_s\big). \tag{2.16}$$

It follows that if Π is a partition of $[0,t]$ and $0 \leqslant s \leqslant t$, then

$$E\big(V_t^{(2)}(M,\Pi) \,\big|\, \mathcal{F}_s\big) = V_s^{(2)}(M,\Pi') + E\big((M_t - M_s)^2 \,\big|\, \mathcal{F}_s\big), \tag{2.17}$$

where Π' is the restriction of Π to $[0,s]$ obtained by inserting s into Π and removing all intervals not included in $[0,s]$. For any partition Π of $\mathbb{R}_+$ and with Π' standing for its restriction to $[0,s]$, set

$$A_s^\Pi := V_s^{(2)}(M,\Pi'). \tag{2.18}$$

For any two partitions Π and $\widetilde{\Pi}$ of $\mathbb{R}_+$ and any $0 \leqslant s \leqslant t$ we have

$$A_s^\Pi - A_s^{\widetilde{\Pi}} = E\big(A_t^\Pi - A_t^{\widetilde{\Pi}} \,\big|\, \mathcal{F}_s\big) \tag{2.19}$$

proving that

$$\big\{A_s^\Pi - A_s^{\widetilde{\Pi}} \colon s \geqslant 0\big\} \text{ is a martingale w.r.t. } \{\mathcal{F}_s\}_{s\geqslant 0} \tag{2.20}$$

Moreover, continuity of M implies that $s \mapsto A_s^\Pi - A_s^{\widetilde{\Pi}}$ is actually continuous as well.

Let Π and $\widetilde{\Pi}$ be two partitions of $\mathbb{R}_+$. In light of (2.20) and continuity, Doob's L^2-maximal inequality (1.13) gives

$$E\Big(\sup_{s\leqslant t} |A_s^\Pi - A_s^{\widetilde{\Pi}}|^2\Big) \leqslant 4E\big(|A_t^\Pi - A_t^{\widetilde{\Pi}}|^2\big). \tag{2.21}$$

The right-hand side is bounded using (2.7). Let $\{\Pi_n\}_{n\geqslant 0}$ be a sequence of partitions of $\mathbb{R}_+$ such that $n \to \|\Pi_n\|$ is non-increasing and

$$E\Big(\operatorname{osc}_M\big([0,n], \|\Pi_n\|\big)^4\Big) \leqslant 64^{-n} \tag{2.22}$$

(This is possible because the $E\big(\operatorname{osc}_M\big([0,n],\delta\big)^4\big) \to 0$ tends to zero as $\delta \to 0$ by the Bounded Convergence Theorem.) The Chebyshev inequality then gives

$$P\Big(\sup_{t\leqslant n} |A_t^\Pi - A_t^{\widetilde{\Pi}}| > 2^{-n}\Big) \leqslant 4 \cdot 112\, K^2 2^{-n} \tag{2.23}$$

Denoting

$$\Omega^\star := \Omega \smallsetminus \Big\{\sup_{t\leqslant n} |A_t^\Pi - A_t^{\widetilde{\Pi}}| > 2^{-n} \text{ i.o.}\Big\} \tag{2.24}$$

Borel-Cantelli lemma gives $P(\Omega^\star) = 1$ and

$$\forall t \geqslant 0\colon\ A_t := \lim_{n\to\infty} A_t^{\Pi_n} \text{ exists on } \Omega^\star \tag{2.25}$$

with the limit locally uniform. On $\Omega \smallsetminus \Omega^\star$, we set $A_t := 0$ for $t \geqslant 0$. Since A^Π is continuous, the fact that the limit is uniform implies that A is continuous.

The inequality (2.7) shows that $V_t^{(2)}(M,\Pi_n) \to A_t$ in probability and L^2 for any sequence $\{\Pi_n\}_{n\geqslant 0}$ of partitions of $[0,t]$ with $\|\Pi_n\| \to 0$. The bound $E(A_t) \leqslant 4K^2$ follows from

$$E\big(V_t^{(2)}(M,\Pi)\big) = E\big((M_t - M_0)^2\big) \leqslant 4K^2 \tag{2.26}$$

and Fatou's lemma. Since $\mathcal{F}_0$ contains all P-null sets, the construction implies that the process A is adapted. For a partition Π of $[0,t]$ containing s, we also have

$$A_t^\Pi - A_s^\Pi = V_t^{(2)}(M,\Pi) - V_s^{(2)}(M,\Pi') \geqslant 0 \tag{2.27}$$

It follows that, for $s \leqslant t$ we have $A_s \leqslant A_t$ a.s. Denoting $\Omega_0 := \bigcap_{s,t\in Q, s<t}\{A_s \leqslant A_t\}$, for each $t \geqslant 0$ we now set

$$\langle M\rangle_t := \begin{cases} A_t, & \text{on } \Omega_0,\\ 0, & \text{else.}\end{cases} \tag{2.28}$$

Then $\langle M\rangle$ is continuous and non-decreasing on rationals and so non-decreasing everywhere. Since $P(\Omega_0) = 1$, the process is $\langle M\rangle$ also adapted and (2.15) remains in force. □

2.3 Proof of existence.

Equipped with above observations, we are now ready to give:

Proof of Theorem 2.1, existence. Let us first assume that $\{M_t : t \geqslant 0\}$ is a bounded martingale, meaning there exists $K \geqslant 0$ such that $\sup_{t\geqslant 0}|M_t| \leqslant K$ a.s. Let $\{\langle M\rangle_t : t \geqslant 0\}$ be the process constructed in Lemma 2.4. Then for each $0 \leqslant s \leqslant t$ and any partition Π containing s,

$$E(M_t^2|\mathcal{F}_s) - M_s^2 = E\big((M_t - M_s)^2\big|\mathcal{F}_s\big) = E\Big(V_t^{(2)}(M,\Pi) - V_s^{(2)}(M,\Pi')\,\Big|\,\mathcal{F}_s\Big) \tag{2.29}$$

where Π' is the restriction of Π to $[0,s]$. Taking $\{\Pi_n\}_{n\geqslant 0}$ with $\|\Pi_n\| \to 0$, the convergence from Corollary 2.4 shows

$$E(M_t^2|\mathcal{F}_s) - M_s^2 = E\big(\langle M\rangle_t - \langle M\rangle_s\,\big|\,\mathcal{F}_s\big) \tag{2.30}$$

which using that $\langle M\rangle$ is adapted shows that $\{M_t^2 - \langle M\rangle_t : t \geqslant 0\}$ is a martingale.

Next assume only that $\{M_t : t \geqslant 0\}$ is a local martingale. Define

$$\tau_K := \inf\{t \geqslant 0 : |M_t| \geqslant K\} \tag{2.31}$$

and let $\langle M\rangle^{(K)}$ be the unique non-decreasing, continuous and adapted process with $\langle M\rangle_0^{(K)} = 0$ such that $\{M_{\tau_K\wedge t}^2 - \langle M\rangle_t^{(K)} : t \geqslant 0\}$ is a martingale. Then for $L \geqslant K$, the process $\{\langle M\rangle_{\tau_K\wedge t}^{(K)} - \langle M\rangle_{\tau_K\wedge t}^{(L)} : t \geqslant 0\}$ is a continuous martingale of bounded variation, implying that it is constant a.s. by Lemma 2.2. Denote

$$\Omega_0 := \{\tau_K \to \infty\} \cap \bigcap_{L\geqslant K}\big\{\forall t\in[0,\tau_K] : \langle M\rangle_t^{(K)} = \langle M\rangle_t^{(L)}\big\} \tag{2.32}$$

and observe that $\lim_{K\to\infty}\langle M\rangle_t^{(K)}$ exists on Ω_0 due to the fact that the sequence is constant once $\tau_K \geqslant t$. We may thus define

$$\langle M\rangle_t := \begin{cases} \lim_{K\to\infty}\langle M\rangle_t^{(K)}, & \text{on } \Omega_0,\\ 0, & \text{else,}\end{cases} \tag{2.33}$$

and observe that $\langle M\rangle$ is non-decreasing and continuous with $\langle M\rangle_0 = 0$.

Since $\tau_n \to \infty$ a.s. by continuity of M, the above observations imply that $P(\Omega_0) = 1$ and so $\langle M\rangle$ is also adapted. The fact that $\langle M\rangle_{\tau_K\wedge t} = \langle M\rangle_t^{(K)}$ for all $t \geqslant 0$ on Ω_0 now shows that $\{M^2_{\tau_K \wedge t} - \langle M \rangle_{\tau_K \wedge t} \colon t \geqslant 0\}$ a martingale for each $K \geqslant 1$. Using $\tau_K \to \infty$ a.s. we conclude that $\{M_t^2 - \langle M \rangle_t \colon t \geqslant 0\}$ is a local martingale, as desired. $\square$

Here are some examples of quadratic variation processes. For M being the standard Brownian motion B, we have $\langle B \rangle_t = t$. Since the Brownian motion is a Gaussian process that has all moments, we conclude that $\{B_t^2 - t \colon t \geqslant 0\}$ is martingale. For the quadratic variation of this martingale, recall that for $M_t := \int_0^t Y_s \mathrm{d}B_s$ we get $\langle M \rangle_t = \int_0^t Y_s^2 \mathrm{d}s$. Hence $\langle B^2 - \mathrm{id} \rangle_t = 4\int_0^t B_s^2 \mathrm{d}s$. Yet another class of examples arises by taking a deterministic continuous strictly increasing function $\phi \colon \mathbb{R}_+ \to \mathbb{R}_+$ and noting that $\{B_{\phi(t)} \colon t \geqslant 0\}$ is a continuous martingale with $\langle B_\phi \rangle_t = \phi(t)$. The point of this example is that neither the martingale nor the quadratic variation process may arise from integrals (which would be the case here only if ϕ is an AC function).

Further reading: Chapter 1 of Karatzas-Shreve