18. Solving the Stochastic Heat Equation

We will now move to applications of the techniques developed in previous chapters, starting with an elementary example and then moving to our first rough differential equation called the Stochastic Heat Equation (SHE). Here is the warm-up example:

18.1 A warm-up example.

Let B be the standard Brownian motion started from zero. Consider the SDE

$$\mathrm{d}X_t = X_t \mathrm{d}B_t \tag{18.1}$$

with initial condition $X_0 = x_0$ for some number $x_0 \in \mathbb{R}$. A strong solution is then a process adapted to the augmented Brownian filtration $\{\widetilde{\mathcal{F}}_t^B\}$. Assuming also that X is square integrable, the Itô chaos decomposition gives

$$X_t = EX_t + \sum_{n\geqslant 1} I_t^{(n)}(f_n) \tag{18.2}$$

for some functions $f_n \in L^{2,\mathrm{loc}}(D_n)$. Note that, under the assumption of square integrability, X is a martingale and so

$$EX_t = EX_0 = x_0 \tag{18.3}$$

Applying the SDE we then get

$$X_t = x_0 + \int_0^t x_0 \,\mathrm{d}B_s + \sum_{n\geqslant 1} I_t^{(n+1)}(f_n 1_{D_{n+1}}) \tag{18.4}$$

where

$$f_n 1_{D_{n+1}}(t_1,\dots,t_{n+1}) := f_n(t_1,\dots,t_n) 1_{\{t_n<t_{n+1}\}}. \tag{18.5}$$

Comparing this with (18.2), the uniqueness of the Itô decomposition implies

$$f_1 = x_0 \tag{18.6}$$

and

$$\forall n \geqslant 1\colon\;\; f_{n+1} = f_n 1_{D_n} \tag{18.7}$$

Using an elementary induction argument, this shows $f_n = x_0 1_{D_n}$ for each $n \geqslant 1$ and so Corollary 17.6 shows

$$X_t = x_0\biggl(1 + \sum_{n\geqslant 1} \frac{1}{n!} t^{n/2} h_n(B_t/\sqrt{t})\biggr) \tag{18.8}$$

The calculation (16.13) then allows us to write this as

$$X_t = x_0\,\mathrm{e}^{B_t - \frac12 t} \tag{18.9}$$

which is of course readily shown to solve the SDE (18.1).

The point of this example was not to solve the SDE (18.1); indeed, the expansion method will hardly be better than methods for finding explicit solutions. The real point is that the expansion is capable of writing a solution to an SDE even if no closed form of the solution exists or can be found using the known methods.

18.2 Stochastic heat equation.

In order to demonstrate the power of expansion method, we will use it to solve our first rough differential equation; namely, the *stochastic heat equation* (SHE)

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \dot W(t,x), & t>0,\ x\in\mathbb{R}^d, \\ u(0,x) = u_0(x), & x\in\mathbb{R}^d \end{cases} \tag{18.10}$$

where $u_0\colon \mathbb{R}^d\to\mathbb{R}$ is the initial condition,

$$\Delta u(t,x) := \sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2}(t,x) \tag{18.11}$$

is the Laplacian in the x variable and $\dot W(t,x)$ is (the space-time derivative of) space-time white noise; i.e., a centered Gaussian process with (formal) covariance structure

$$\operatorname{Cov}\bigl(\dot W(t,x),\dot W(t',x')\bigr) = \delta(t-t')\delta(x-x') \tag{18.12}$$

We are interested in the full-space Cauchy problem; i.e., a solution $u\colon\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}$ subject to only the initial condition $u(0,x)=u_0(x)$ but otherwise no *a priori* restriction on the behavior as $|x|\to\infty$.

The above problem is not really well posed due to the fact that $\dot W(t,x)$ is not (and cannot be) defined as a function. In addition, an interpretation of what it means to "solve" the above equation is necessary. Indeed, interpreting the equation in integral form does not help either because even $\int_0^t \dot W(x,s)\mathrm{d}s$ is a singular object. (The integrals for different x are independent and equidistributed.)

A mathematically meaningful interpretation of (18.10) relies on the concept of a weak solution. In the present case this amounts to the following:

Definition 18.1 *A process $\{u(t,x)\colon t\geqslant 0, x\in\mathbb{R}^d\}$ is a weak solution to* (18.10) *if*

$$\int\Bigl(-\frac{\partial h}{\partial t}(t,x) - \Delta h(t,x)\Bigr)u(t,x)\mathrm{d}t\mathrm{d}x = \int h(t,x)W(\mathrm{d}t\mathrm{d}x) \tag{18.13}$$

holds a.s. for each C^2-function $h\colon\mathbb{R}^+\times\mathbb{R}^d\to\mathbb{R}$ with compact support.

Here the expression on the left is obtained by integrating the PDE against h and formally commuting all the derivatives to h. The expression on the right is the rigorous version of the integral $\int h(t,x)\dot W(t,x)\mathrm{d}t\mathrm{d}x$.

In order to produce a weak solution, we will rely on approximation. Specifically, we replace the ill-defined term $\dot W$ by a continuous smoothed-out process $\dot W_\epsilon$ that obeys

$$\int h(t,x)\dot W_\epsilon(t,x)\mathrm{d}t\,\mathrm{d}x \ \underset{\epsilon\downarrow 0}{\overset{P}{\longrightarrow}}\ \int h(t,x)W(\mathrm{d}t\mathrm{d}x). \tag{18.14}$$

for all h bounded, measurable with compact support. This makes the problem (18.10) meaningful in the classical sense. In particular, as we will show, the smoothed-out problem admits a unique solution u^ϵ with sub-Gaussian growth as $\|x\|\to\infty$. We then aim to prove that u^ϵ converges weakly to some u as $\epsilon\downarrow 0$. This u will obey (18.13).

The smoothed-out version of the white noise will be defined using a *mollifier* φ. This is a continuous function $\varphi\colon \mathbb{R}_+ \times \mathbb{R}^d \to [0,\infty)$ with compact support subject to integral constraint $\int_{\mathbb{R}_+\times\mathbb{R}^d} \varphi(t,x)\mathrm{d}t\mathrm{d}x = 1$. We will scale space and time by ϵ using

$$\varphi_\epsilon(t,x) := \frac{1}{\epsilon^{d+1}}\varphi\big(\epsilon^{-1}t, \epsilon^{-1}x\big) \tag{18.15}$$

where the normalization ensures that $\int_{\mathbb{R}_+\times\mathbb{R}^d} \varphi_\epsilon(t,x)\mathrm{d}t\mathrm{d}x = 1$ remains in force for each $\epsilon > 0$. Then we set

$$\dot W_\epsilon(t,x) = \varphi_\epsilon * \dot W(t,x) := \int \varphi_\epsilon(t,x) W(\mathrm{d}t\mathrm{d}x) \tag{18.16}$$

where the right-hand side is meaningful due to the fact that $\varphi_\epsilon \in L^2(\mathbb{R}_+ \times \mathbb{R}^d)$.

Lemma 18.2 *Suppose that φ is γ-Hölder continuous for some $\gamma \in (0,1]$. Then for all $\epsilon \in (0,1)$ and all $\gamma' \in (0,\gamma)$, the process*

$$\big\{\dot W_\epsilon(t,x)\colon t \geqslant 0,\ x \in \mathbb{Z}^d\big\} \tag{18.17}$$

admits a locally γ'-Hölder continuous version such that for all bounded measurable $h\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ with compact support

$$\int h(t,x)\dot W_\epsilon(t,x)\mathrm{d}t\mathrm{d}x = \int h * \varphi_\epsilon \,\mathrm{d}W. \tag{18.18}$$

In particular, (18.14) *holds for all bounded measurable $h\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ with compact support.*

Proof. Suppose φ is γ-Hölder continuous. For $x, x' \in \mathbb{R}^d$ and $t, t' \geqslant 0$, the isometry underlying the definition of the Paley-Wiener integral gives

$$E\Big(\big|\dot W_\epsilon(t,x) - \dot W_\epsilon(t',x')\big|^2\Big) = \int \big|\varphi_\epsilon(x-y, t-s) - \varphi_\epsilon(x'-y, t'-s)\big|^2 \mathrm{d}s\,\mathrm{d}y. \tag{18.19}$$

The Hölder continuity along with the fact that the integrand has compact support implies that the right-hand side is bounded by $C(\epsilon)(|t-t'|^{2\gamma} + \|x-x'\|^{2\gamma})$ for some ϵ-dependent constant $C(\epsilon)$. Since $\dot W_\epsilon(t,x) - \dot W_\epsilon(t',x')$ is centered Gaussian, the fact that the $2n$-th moment of a centered Gaussian is bounded by an n-dependent constant times n-th power of the variance shows

$$E\Big(\big|\dot W_\epsilon(t,x) - \dot W_\epsilon(t',x')\big|^{2n}\Big) \leqslant C_n(\epsilon)\big(|t-t'|^{2n\gamma} + \|x-x'\|^{2n\gamma}\big) \tag{18.20}$$

Writing $\alpha_n := 2n\gamma$ and $\beta_n := 2n\gamma - (d+1)$, the Kolmogorov-Čenstov Theorem tells us that $\dot W_\epsilon$ admits a locally γ'-Hölder continuous version for each $\gamma' < \beta_n/\alpha_n$. Since $\beta_n/\alpha_n \to \gamma$, we get the first part of the claim.

The second part follows from the following Fubini theorem for mixed Paley-Wiener and ordinary Lebesgue integral:

Lemma 18.3 *Let W be white noise on finite measure space $(\mathscr{X}, \mathcal{G}, \mu)$ and let $(\mathscr{Y}, \mathcal{H}, \nu)$ be a finite measure space. Then for all $f \in L^2(\mathscr{X} \times \mathscr{Y}, \mathcal{G} \otimes \mathcal{H}, \mu \otimes \nu)$ bounded measurable the map*

$y \mapsto \int f(\cdot,y)\mathrm{d}W$ admits a measurable version and for all $g \in L^\infty(\mathscr{Y},\mathcal{H},\nu)$,

$$\int g(y)\biggl(\int f(\cdot,y)\mathrm{d}W\biggr)\nu(\mathrm{d}y) = \int\biggl(\int f(\cdot,y)g(y)\nu(\mathrm{d}y)\biggr)\mathrm{d}W \tag{18.21}$$

holds a.s. with the integral w.r.t. ν on the left converging absolutely a.s.

Leaving the proof of this lemma to homework exercise, we now move the last part of the claim. Fix a bounded measurable function $h\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ with compact support. Then (18.18) along with the Paley-Wiener isometry give

$$\begin{aligned} &E\biggl(\Bigl|\int h(t,x)\dot{W}_\epsilon(t,x)\mathrm{d}t\,\mathrm{d}x - \int h\,\mathrm{d}W\Bigr|^2\biggr)\\ &\qquad= E\biggl(\Bigl|\int h*\varphi_\epsilon(t,x)\,\mathrm{d}W - \int h\,\mathrm{d}W\Bigr|^2\biggr) = \int\bigl|h*\varphi_\epsilon(t,x) - h(t,x)\bigr|^2\mathrm{d}t\,\mathrm{d}x. \end{aligned} \tag{18.22}$$

Since h is locally integrable, the Lebesgue differentiation theorem shows $h*\varphi_\epsilon \to h$ a.e. as $\epsilon \downarrow 0$. The boundedness of h gives uniform boundedness of $h*\varphi_\epsilon$ and if h is supported in $[-n,n]^d$ then $h*\varphi_\epsilon$ is supported in $[-n-k,n+k]$ where k is such that φ is supported in $[-k,k]^d$ for all $\epsilon \in (0,1)$. The right-hand side of (18.22) thus tends to zero as $\epsilon \downarrow 0$ by the Bounded Convergence Theorem, proving (18.14). □

18.3 Solving the SHE perturbatively.

With the noise smoothed-out, we now invoke the following general fact:

Lemma 18.4 *Let $f\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ be continuous and let $u_0\colon \mathbb{R}^d \to \mathbb{R}$ be continuous such that, for some $\eta \in (0,2)$ and all $t \geqslant 0$,*

$$\sup_{x\in\mathbb{R}^d}\frac{\log(|u_0(x)|\vee 1)}{1+\|x\|^\eta} < \infty \;\wedge\; \sup_{s\leqslant t}\sup_{x\in\mathbb{R}^d}\frac{\log(|f(s,x)|\vee 1)}{1+\|x\|^\eta} < \infty \tag{18.23}$$

Denote $g_t(x) := 1_{[0,\infty)}(t)(4\pi t)^{-d/2}\mathrm{e}^{-\frac{\|x\|^2}{4t}}$. Then

$$u(t,x) := g_t * u_0(x) + \int_{[0,t]\times\mathbb{R}^d} g_{t-s}(x-y)f(s,y)\mathrm{d}s\,\mathrm{d}y \tag{18.24}$$

is well defined, is of type C^1/C^2 and solves

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + f(t,x), & t>0,\ x\in\mathbb{R}^d,\\ u(0,x) = u_0(x), & x\in\mathbb{R}^d. \end{cases} \tag{18.25}$$

Moreover, u obeys

$$\sup_{s\leqslant t}\sup_{x\in\mathbb{R}^d}\frac{\log(|u(s,x)|\vee 1)}{1+\|x\|^\eta} < \infty \tag{18.26}$$

for some $\eta \in (0,2)$ and all $t \geqslant 0$ and the unique function in the class of C^1/C^2-functions satisfying (18.25–18.26).

Proof (key steps). That (18.24) is well defined follows from the subgaussian growth of u_0 and f. This allows us to even differentiate under the integrals and check (using that $t,x \mapsto g_t(x)$ is the fundamental solution of the heat equation) that u solves (18.25).

For uniqueness we note that if u is a solution, then (for B the standard Brownian motion) the Itô formula shows that

$$M_s := u(t-s,\sqrt{2}\,B_s) + \int_0^s f(t-r,\sqrt{2}\,B_r)\mathrm{d}r \tag{18.27}$$

is a local martingale. The bound (18.26) now gives that

$$|M_s| \leqslant (1+t)\exp\Bigl\{C(t)\sup_{s\leqslant t}(1+|B_s|^\eta)\Bigr\} \tag{18.28}$$

for $C(t)$ being the supremum in (18.26) and that on the right of (18.23) which using the fact that $\sup_{s\leqslant t}|B_s|$ has a Gaussian tail is enough to prove that M is a martingale. But then Fubini-Tonelli gives

$$u(t,x) = E^x M_0 = E^x M_t = E^x u(0,\sqrt{2}\,B_t) + \int_0^t E^x\bigl(f(t-r,\sqrt{2}\,B_r)\bigr)\mathrm{d}r \tag{18.29}$$

which by writing the expectation using the probability density of $\sqrt{2}\,B_r$ reduces to the right-hand side of (18.24). □

In order to apply this to the smoothed-out PDE (18.10), we need:

Lemma 18.5 *For all $k \geqslant 1$, all $\epsilon > 0$ and all $t \geqslant 0$,*

$$\sup_{s\leqslant t}\sup_{x\in\mathbb{R}^d}\frac{|\dot W_\epsilon(s,x)|}{1+\|x\|^k} < \infty \quad \text{a.s.} \tag{18.30}$$

Proof (sketch). The argument in the proof of existence of a continuous version via the Kolmogorov-Čenstov Theorem actually shows that $\max_{x\in[0,1]^d}\sup_{s\leqslant t}|\dot W_\epsilon(s,x)|$ has all moments. Since the law of this random variable is invariant under the shifts, the usual argument shows that $s,x \mapsto |\dot W_\epsilon(s,x)|$ grows slower than any polynomial. □

Assume that u_0 obeys the condition in (18.23). On the event that the inequality (18.30) holds, Lemma 18.4 shows existence of a unique function $u^\epsilon\colon \mathbb{R}_+\times\mathbb{R}^d \to \mathbb{R}$ of type C^1/C^2, subject to the growth restriction (18.26) and solving

$$\begin{cases} \dfrac{\partial u^\epsilon}{\partial t}(t,x) = \Delta u^\epsilon(t,x) + \dot W_\epsilon(t,x), & t>0,\ x\in\mathbb{R}^d, \\ u^\epsilon(0,x) = u_0(x), & x\in\mathbb{R}^d. \end{cases} \tag{18.31}$$

In light of (18.24) and (18.18), the function also admits the representation

$$u^\epsilon(t,x) = g_t * u_0(x) + \int\biggl(\int g_{t-s}(x-y)\varphi_\epsilon(s-r,y-z)\mathrm{d}s\mathrm{d}y\biggr)W(\mathrm{d}r\mathrm{d}z). \tag{18.32}$$

We will now examine the limit of this expression as $\epsilon \downarrow 0$. First note that, by the definition of φ_ϵ and the continuity of $t, x \mapsto g_t(x)$, we have

$$\forall t > r > 0\,\forall x, z \in \mathbb{R}^d\colon\quad \int g_{t-s}(x-y)\varphi_\epsilon(s-r, y-z)\mathrm{d}s\mathrm{d}y \underset{\epsilon\downarrow 0}{\longrightarrow} g_{t-r}(x-z) \tag{18.33}$$

The right-hand side is singular at a single point; namely, $(t,x) = (r,z)$ where the left-hand side equals zero while the right-hand size diverges. That being said, what is relevant for (18.32) is L^2-convergence and the question whether the function on the right is in L^2. Here we observe:

Lemma 18.6 *For all $t > 0$ and all $x \in \mathbb{R}^d$,*

$$\int g_{t-r}(x-z)^2\mathrm{d}r\mathrm{d}z \begin{cases} < \infty, & \text{if } d = 1,\\ = \infty, & \text{if } d \geqslant 2.\end{cases} \tag{18.34}$$

Proof. The explicit nature of g_t gives

$$\int g_t(x-z)^2\mathrm{d}z = \frac{1}{(4\pi t)^d}\int \mathrm{e}^{-\frac{\|x-z\|^2}{2t}} = \frac{(2\pi t)^{d/2}}{(4\pi t)^d} = (8\pi)^{-d/2}t^{-d/2} \tag{18.35}$$

The function $t \mapsto t^{-d/2}$ is integrable in $d = 1$ and non-integrable in $d \geqslant 2$. $\square$

We are now ready to state and prove:

Theorem 18.7 (Solution of SHE in $d = 1$) *Suppose $d = 1$ and let $u_0\colon \mathbb{R}^d \to \mathbb{R}$ be such that the first inequality in* (18.23) *holds. Then*

$$u(t,x) := g_t * u_0(x) + \int g_{t-r}(x-z)W(\mathrm{d}r\mathrm{d}z) \tag{18.36}$$

is well defined and admits a continuous version that is a weak solution to (18.10) *in the sense of Definition 18.1. Moreover, any other continuous solution satisfying* (18.23) *a.s. is equal to u except on an event of zero probability.*

Proof. Suppose $d = 1$. The existence of the Paley-Wiener integral follows from Lemma 18.6 so the key issue is to prove existence of a continuous version. Denote the integral by $\bar u(t,x)$ and observe that $\{\bar u(t,x)\colon t \geqslant 0,\ x \in \mathbb{R}\}$ is a mean-zero Gaussian process with

$$\begin{aligned}\operatorname{Cov}\bigl(\bar u(t,x), \bar u(t',x')\bigr) &= \int_0^{t\wedge t'}\mathrm{d}r\int_{\mathbb{R}}\mathrm{d}z g_{t-r}(x-z)g_{t-r'}(x'-z)\\ &= \frac{1}{4\pi}\int_0^{t\wedge t'}\mathrm{d}r\int_{\mathbb{R}}\mathrm{d}z\,\frac{1}{\sqrt{t-r}\sqrt{t'-r}}\mathrm{e}^{-\frac{|x-z|^2}{4(t-r)}-\frac{|x'-z|^2}{4(t'-r)}}.\end{aligned} \tag{18.37}$$

For the exponent we get

$$\begin{aligned}\frac{|x-z|^2}{t-r} &+ \frac{|x'-z|^2}{t'-r} \\ &= \Big(\frac{1}{t-r}+\frac{1}{t'-r}\Big)|z|^2 - 2\Big(\frac{x}{t-r}+\frac{x'}{t'-r}\Big)z + \Big(\frac{|x|^2}{t-r}+\frac{|x'|^2}{t'-r}\Big) \\ &= \Big(\frac{1}{t-r}+\frac{1}{t'-r}\Big)\left(z-\frac{\frac{x}{t-r}+\frac{x'}{t'-r}}{\frac{1}{t-r}+\frac{1}{t'-r}}\right)^2 + \frac{|x-x'|^2}{t+t'-2r}.\end{aligned} \tag{18.38}$$

The integral with respect to z is now performed with the result

$$\begin{aligned}\operatorname{Cov}\big(\bar u(t,x),\bar u(t',x')\big) &= \frac{1}{\sqrt{4\pi}}\int_0^{t\wedge t'}\frac{1}{\sqrt{t+t'-2r}}\mathrm{e}^{-\frac14\frac{|x-x'|^2}{t+t'-2r}}\,\mathrm{d}r \\ &= \frac{1}{2\sqrt{4\pi}}\int_{|t-t'|}^{t+t'}\frac{1}{\sqrt{s}}\,\mathrm{e}^{-\frac{|x-x'|^2}{4s}}\,\mathrm{d}s,\end{aligned} \tag{18.39}$$

where in the second line we substituted $s := t+t'-2r$. This now shows

$$\begin{aligned}E\big(|\bar u(t,x)-\bar u(t',x)|^2\big) &= \frac{1}{4\sqrt{\pi}}\Big(\int_0^{2t}\frac{\mathrm{d}s}{\sqrt{s}}+\int_0^{2t'}\frac{\mathrm{d}s}{\sqrt{s}}-2\int_{|t-t'|}^{t+t'}\frac{\mathrm{d}s}{\sqrt{s}}\Big) \\ = \frac{1}{2\sqrt{\pi}}\int_0^{|t-t'|}\frac{\mathrm{d}s}{\sqrt{s}} &+ \frac{1}{2\sqrt{\pi}}\int_{t\wedge t'}^{t\vee t'}\frac{\mathrm{d}s}{\sqrt{s}} \leqslant \frac{1}{\sqrt{\pi}}\int_0^{|t-t'|}\frac{\mathrm{d}s}{\sqrt{s}} = \frac{2}{\sqrt{\pi}}|t-t'|^{1/2},\end{aligned} \tag{18.40}$$

where we used that $s\mapsto\frac{1}{\sqrt{s}}$ is decreasing to merge the two integrals into one, as a bound. A very similar calculation shows

$$\begin{aligned}E\big(|\bar u(t,x)-\bar u(t,x')|^2\big) &= \frac{1}{2\sqrt{\pi}}\int_0^{2t}\frac{1}{\sqrt{s}}\Big(1-\mathrm{e}^{-\frac{|x-x'|^2}{4s}}\Big)\mathrm{d}s \\ \leqslant \frac{1}{2\sqrt{\pi}}\int_0^{|x-x'|^2}\frac{\mathrm{d}s}{\sqrt{s}} &+ \frac{1}{2\sqrt{\pi}}\int_{|x-x'|^2}\frac{\mathrm{d}s}{s^{3/2}}|x-x'|^2 = \frac{2}{\sqrt{\pi}}|x-x'|.\end{aligned} \tag{18.41}$$

This gives that, as far as second moments are concerned, the process behaves as 1/4-Hölder in time and 1/2-Hölder is space. Being Gaussian, the corresponding bounds extend to all moments which, following the argument used in Lemma 18.2, proves the existence of a continuous version that is locally γ'-Hölder in t for each $\gamma' < 1/4$ and locally γ''-Hölder in x, for each $\gamma'' < 1/2$.

It remains to prove that u is the unique weak solution. Let $h \in C^2(\mathbb{R}_+\times\mathbb{R}^d)$ have compact support. Then Lemma 18.3 gives

$$\begin{aligned}\int\Big(-\frac{\partial h}{\partial t}(t,x)-\Delta h(t,x)\Big)u(t,x)\mathrm{d}t\mathrm{d}x &= \int\Big(-\frac{\partial h}{\partial t}(t,x)-\Delta h(t,x)\Big)g_t * u_0(x)\mathrm{d}t\mathrm{d}x \\ &+ \int\bigg(\int\Big(-\frac{\partial h}{\partial t}(t,x)-\Delta h(t,x)\Big)g_{t-s}(x-y)\mathrm{d}t\mathrm{d}x\bigg)W(\mathrm{d}s\mathrm{d}y)\end{aligned} \tag{18.42}$$

The fact that g_t is the fundamental solution and thus that $t,x\mapsto g_t(x)$ solves the heat equation implies the first integral on the right vanishes. Commuting the derivatives in the inner integral in the second term shows

$$\int\Bigl(-\frac{\partial h}{\partial t}(t,x)-\Delta h(t,x)\Bigr)g_{t-s}(x-y)\mathrm{d}t\mathrm{d}x = h(x,y) \tag{18.43}$$

proving that (18.13) holds. For uniqueness, note that if $\tilde u$ is another weak solution, then $u-\tilde u$ is a weak solution to the heat equation with zero initial data. But this is known to vanish in the class of functions with subgaussian growth using similar arguments as invoked above. □

The reader may wonder why we have not shown that u^ϵ actually converges to u. This is true albeit requires computations that we prefer to defer to a homework assignment at this point. What is perhaps more interesting is the question what happens in $d\geqslant 2$. Here the following applies:

Theorem 18.8 *Let $d\geqslant 1$ and let u^ϵ is the solution to* (18.10) *with smoothed-out noise and initial date subject to* (18.23)*. Then there exists a Gaussian process $\{X_h\colon h\in C_c(\mathbb{R}_+\times\mathbb{R}^d)\}$ with mean zero and*

$$\operatorname{Cov}(X_g,X_h)=\int C_d(t,x,t',x')g(t,x)h(t',x')\mathrm{d}t\mathrm{d}x\mathrm{d}t'\mathrm{d}x', \tag{18.44}$$

where

$$C_d(t,x,t',x'):=\frac{1}{2(4\pi)^{d/2}}\int_{|t-t'|}^{t+t'}\frac{1}{s^{d/2}}\,\mathrm{e}^{-\frac{\|x-x'\|^2}{4s}}\mathrm{d}s, \tag{18.45}$$

such that

$$\forall h\in C_c(\mathbb{R}_+\times\mathbb{R}^d)\colon\quad \int h(t,x)u^\epsilon(t,x)\mathrm{d}t\mathrm{d}x \ \underset{\epsilon\downarrow 0}{\overset{P}{\longrightarrow}}\ X_h \tag{18.46}$$

Moreover, $h\mapsto X_h$ is linear.

We leave the proof of this to the reader while noting that (18.45) is a refinement of (18.39) in $d\geqslant 2$. The limit process is thus no longer a function but rather a linear *functional* on the space of continuous functions with compact support. (The functional is actually continuous albeit with respect to a topology that does not permit interpreting it as a function in $d\geqslant 2$.) To understand the covariances, note that

$$C_d(t,x,t,x')\sim\begin{cases}\log\|x-x'\|, & \text{if } d=2,\\ \|x-x'\|^{2-d}, & \text{if } d\geqslant 3,\end{cases} \tag{18.47}$$

This is the behavior known from a process called *Gaussian Free Field*. The time-correlations are actually quite bad already in $d=1$ where they lead to a process that is not even $1/4$-Hölder continuous. This is much rougher than anything we have seen so far; indeed, all diffusions are γ-Hölder for $\gamma<1/2$. This is our first encounter with limitations of Itô's approach to stochastic processes.