17. Itô chaos decomposition

Here we introduce the iterated Itô integrals as particular form of the higher-order Paley-Wiener integrals for the white noise associated with the standard Brownian motion. Then we prove the corresponding form of the chaos expansion.

17.1 Iterated Itô integrals.

An iterated Itô integral is an expression of the form

$$\int_0^t \Bigl(\int_0^{t_n} \Bigl(\dots \Bigl(\int_0^{t_2} f(t_1,\dots,t_n)\mathrm{d}B_{t_1}\Bigr)\dots\Bigr)\mathrm{d}B_{t_{n-1}}\Bigr)\mathrm{d}B_{t_n} \tag{17.1}$$

where $f\colon [0,\infty)^n \to \mathbb{R}$ is a function with suitable integrability properties. To define this precisely, we start with some notations.

Noting that the integral (17.1) "sees" only the arguments of f where $t_1 < t_2 \dots < t_n$, the function really only needs to be defined on the set

$$D_n := \bigl\{(t_1,\dots,t_n) \in \mathbb{R}^n \colon 0 \leqslant t_1 < t_2 < \dots < t_n\bigr\} \tag{17.2}$$

We endow D_n with the n-dimensional Lebesgue measure and write $L^{2,\mathrm{loc}}(D_n)$ for the space of locally square-integrable functions $f\colon D_n \to \mathbb{R}$. For $f\colon D_n \to \mathbb{R}$ and $t \geqslant 0$, let $f_t\colon D_{n-1} \to \mathbb{R}$ denote the function

$$f_t(t_1,\dots,t_{n-1}) := 1_{\{t_{n-1}<t\}} f(t_1,\dots,t_{n-1},t) \tag{17.3}$$

We will henceforth assume existence of a probability space supporting a Brownian motion B adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ with $\mathcal{F}_0$ containing all P-null sets. Recall also that $\mathcal{V}_B$ is the space of adapted, jointly-measurable processes $\{Y_s\colon s \geqslant 0\}$ such that $s,\omega \mapsto Y_s(\omega)$ is in $L^2([0,t]\times\Omega)$ for all $t \geqslant 0$. We then have:

Proposition 17.1 (Iterated Itô integrals) *For all $n \geqslant 1$ and all $f \in L^{2,\mathrm{loc}}(D_n)$ there exists a continuous L^2-martingale $\{I_t^{(n)}(f)\colon t \geqslant 0\}$ with $I_0^{(n)}(f) = 0$ such that*

$$\forall f \in L^{2,\mathrm{loc}}(D_1)\,\forall t \geqslant 0\colon\;\; I_t^{(1)}(f) = \int_0^t f(s)\mathrm{d}B_s \quad \text{a.s.} \tag{17.4}$$

and such that for all $n \geqslant 2$ and $f \in L^2(D_n)$ there exists $Y \in \mathcal{V}_B$ with the property that

$$\forall t \geqslant 0\colon \quad Y_t = I_t^{(n-1)}(f_t) \quad \text{a.s.} \tag{17.5}$$

and

$$\forall t \geqslant 0\colon \quad I_t^{(n)}(f) = \int_0^t Y_s\,\mathrm{d}B_s \quad \text{a.s.} \tag{17.6}$$

Moreover, for each $t \geqslant 0$, the map $f \mapsto I_t^{(n)}(f)$ obeys

$$E\bigl(I_t^{(n)}(f)^2\bigr) = \int_{D_n\cap[0,t]^n} f(t_1,\dots,t_n)^2\,\mathrm{d}t_1\dots\mathrm{d}t_n \tag{17.7}$$

and so defines a linear isometry $L^2(D_n\cap[0,t]^n) \to L^2(\Omega,\mathcal{F}^B,P)$.

Before we delve into the proof, note that the statements allows us to put

$$\int_0^t \Bigl(\int_0^{t_n} \Bigl(\dots \Bigl(\int_0^{t_2} f(t_1,\dots,t_n)\mathrm{d}B_{t_1}\Bigr)\dots\Bigr)\mathrm{d}B_{t_{n-1}}\Bigr)\mathrm{d}B_{t_n} := I_t^{(n)}(f). \tag{17.8}$$

The conditions (17.5–17.6) express the nesting property of these integrals which is intuitive but, since these are not ordinary integrals, has to be handled with care. We state the nesting this way because we do not want to deal with the regularity of $t \mapsto I_t^{(n-1)}(f)$. This explanation should be enough for us to get into:

Proof of Proposition 17.1. We proceed as in the construction of the Itô integral. First, let us call $f\colon D_n \to \mathbb{R}$ simple if for some $m \geqslant n$ and $0 \leqslant s_1 < \dots < s_m$ and some collection of numbers $\{a_{j_1,\dots,j_n}\colon 1 \leqslant j_1 < j_2 < \dots < j_n \leqslant m\} \subseteq \mathbb{R}$,

$$f(t_1,\dots,t_n) = \sum_{1\leqslant j_1<j_2<\dots<j_n\leqslant m} a_{j_1,\dots,j_n} \prod_{k=1}^n 1_{(s_{j_k-1},s_{j_k}]}(t_k) \tag{17.9}$$

holds for all $(t_1,\dots,t_n) \in D_n$. We then define $t \mapsto I_t^{(n)}(f)$ as

$$I_t^{(n)}(f) := \sum_{1\leqslant j_1<j_2<\dots<j_n\leqslant m} a_{j_1,\dots,j_n} \prod_{k=1}^n \bigl(B_{s_{j_k}\wedge t} - B_{s_{j_k-1}\wedge t}\bigr) \tag{17.10}$$

which requires checking (left to the reader) that the right-hand side does not depend on the representation of f as above.

Note that for $n = 1$ this is exactly the definition of the Itô integral of a simple function. Observe also that, for $n \geqslant 2$, we have

$$I_t^{(n-1)}(f_t) = \sum_{1\leqslant j_1<j_2<\dots<j_n\leqslant m} a_{j_1,\dots,j_n} 1_{(s_{j_n-1},s_{j_n}]}(t) \prod_{k=1}^{n-1} \bigl(B_{s_{j_k}\wedge t} - B_{s_{j_k-1}\wedge t}\bigr) \tag{17.11}$$

which is checked to be adapted and piecewise constant (as a process indexed by t). Setting $Y_t := I_t^{(n-1)}(f_t)$ we observe that Y is simple (i.e., $Y \in \mathcal{V}_0$) and that (17.6) is in force. The isometry and the continuous-martingale property are then checked readily as well.

The claim thus holds for f simple so the main piece of work is to extend it to all $f \in L^{2,\mathrm{loc}}(D_n)$. For this we observe:

Lemma 17.2 *Simple functions of the form* (17.9) *are dense in* $L^{2,\mathrm{loc}}(D_n)$.

Proof. It suffices to show that if $h \in L^2(D_n)$ with compact support is orthogonal to all simple functions, then $h = 0$. Given $0 \leqslant a_1 < b_1 \leqslant a_2 < b_2 \leqslant \dots \leqslant a_n < b_n$, the function

$$f(t_1,\dots,t_n) := \prod_{i=1}^n 1_{(a_i,b_i]}(t_i) \tag{17.12}$$

is simple. The orthogonality of h to f then shows that the integral of h vanishes on all sets of the form $\times_{i=1}^n (a_i,b_i]$ for $\{a_i,b_i\}_{i=1}^n$ as above. But these sets form a semialgebra (and thus a π-system) that generates all Borel subsets of D_n. Fix any $t \geqslant 0$. Since the class of Borel subsets of $D_n \cap [0,t]^n$ on which the integral of h vanishes forms a λ-system, Dynkin's π/λ-Theorem shows that the integral of h vanishes on all Borel subsets of $D_n \cap [0,t]^n$. This implies $h = 0$ Lebesgue a.e. as desired. $\square$

Continuing the proof of Proposition 17.1, Lemma 17.2 allows us to approximate any $f \in L^{2,\mathrm{loc}}(D_n)$ by a sequence of simple functions $\{f^{(k)}\}_{k\geqslant 1}$ so that

$$\|f - f^{(k)}\|_{L^{2,\mathrm{loc}}(D_n)} \leqslant 4^{-k} \tag{17.13}$$

for each $k \geqslant 1$. Noting that (17.7) for $n-1$ instead of n gives

$$\int_0^t E\Bigl(\bigl[I_s^{(n-1)}(f_s^{(k+1)}) - I_s^{(n-1)}(f_s^{(k)})\bigr]^2\Bigr)\mathrm{d}s \leqslant \|f^{(k+1)} - f^{(k)}\|^2_{L^{2,\mathrm{loc}}(D_n)} \leqslant 16^{1-k} \tag{17.14}$$

the Markov inequality shows

$$P\biggl(\lambda\Bigl(\bigl\{s \in [0,t] \colon |I_s^{(n-1)}(f_s^{(k+1)}) - I_s^{(n-1)}(f_s^{(k)})| > 2^{-k}\bigr\}\Bigr) > 2^{-k}\biggr) \leqslant 8^k 16^{1-k} \tag{17.15}$$

Write $\Omega^\star$ for the event that the event that the event in the probability occurs only for finitely-many k. Setting

$$Y_t := \limsup_{k\to\infty} I_t^{(n-1)}(f_t^{(k)}) 1_{\{\limsup_{k\to\infty} I_t^{(n-1)}(f_t^{(k)}) \in \mathbb{R}\}} \tag{17.16}$$

on $\Omega^\star$ and $Y_t := 0$ on $\Omega \smallsetminus \Omega^\star$, then the limit in

$$Y_t := \lim_{k\to\infty} I_t^{(n-1)}(f_t^{(k)}) \tag{17.17}$$

exists and equality holds for Lebesgue a.e. $t \in [0,\infty)$ on $\Omega^\star$.

The Borel-Cantelli lemma implies that $P(\Omega^\star) = 1$ and the fact $\mathcal{F}_0$ contains all P-null sets gives $\Omega^\star \in \mathcal{F}_0$. The process Y is then jointly measurable and adapted. The inequality (17.14) with the help of Fatou's lemma also shows that

$$\int_0^t E\Bigl(\bigl[I_s^{(n-1)}(f_s^{(k)}) - Y_s\bigr]^2\Bigr) \underset{k\to\infty}{\longrightarrow} 0 \tag{17.18}$$

giving us (17.5). The convergence also implies $Y \in \mathcal{V}_B$ and the Itô integral in (17.6) is well defined and equal to the L^2-limit of the integrals in

$$I_t^{(n)}(f^{(k)}) = \int_0^t I_s^{(n-1)}(f_s^{(k)})\mathrm{d}B_s. \tag{17.19}$$

Using again that $\mathcal{F}_0$ contains all P-null set, we now define $\{I_t^{(n)}(f) \colon t \geqslant 0\}$ to be a continuous version of $\{\int_0^t Y_s \mathrm{d}B_s \colon t \geqslant 0\}$. Then (17.6) holds and the process is a continuous L^2-martingale as claimed. Validating also the isometry (17.7) by extension from simple functions, the proof is finished. $\square$

17.2 Chaos decomposition.

In order to connect with the topic discussed above, we now observe:

Theorem 17.3 *Let $B = \{B_s \colon s \in [0,t]\}$ be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. Fix $t \geqslant 0$ and let $\widetilde{\mathcal{F}}_t^B := \sigma(\mathcal{N} \cup \sigma(B_s \colon s \leqslant t))$ for $\mathcal{N}$ being the P-null sets. Let $\{\mathcal{H}_n\}_{n \geqslant 0}$ be the closed linear subspaces of $L^2(\Omega, \widetilde{\mathcal{F}}_t^B, P)$ constructed via (14.9–14.10). Then*

$$\forall n \geqslant 1\colon \quad \mathcal{H}_n = \Big\{ I_t^{(n)}(f) \colon f \in L^2(D_n \cap [0,t]^n) \Big\} \tag{17.20}$$

Proof. Let $\widetilde{\mathcal{H}}_n$ denote the set on the right of (17.20). Thanks to the isometry (17.7) (and the construction of iterated Itô integrals by L^2-limits of those of simple processes), $\widetilde{\mathcal{H}}_n$ is a closed linear subspace of $L^2(\Omega, \widetilde{\mathcal{F}}_t^B, P)$. Next note that (17.6) along with the Itô isometry give, for any $f \in L^2(D_n \cap [0,t]^n)$ and $g \in L^2(D_m \cap [0,t]^m)$, that

$$E\Big(I_t^{(n)}(f) I_t^{(m)}(g) \Big) = \int_0^t E\Big(I_s^{(n-1)}(f_s) I_s^{(m-1)}(g_s) \Big) \mathrm{d}s \tag{17.21}$$

The right-hand side vanishes for the case $n > 1$ and $m = 1$ by the fact that Itô integral is centered. By induction, $\widetilde{\mathcal{H}}_n \perp \widetilde{\mathcal{H}}_m$ whenever $m \neq n$.

Denote $\widetilde{\mathcal{H}}_0 := \mathcal{H}_0$. In light of the closedness of $\bigoplus_{k=0}^n \mathcal{H}_k$ and the integrals of simple functions being linear combinations of products of n Brownian increments, see (17.10), we have $I_t^{(n)}(f) \in \bigoplus_{i=0}^n \mathcal{H}_k$ for each $f \in L^2(D_n \cap [0,t]^n)$. Hence, $\bigoplus_{k=0}^n \widetilde{\mathcal{H}}_k \subseteq \bigoplus_{k=0}^n \mathcal{H}_k$. It now suffices to show

$$\forall n \geqslant 0\colon \quad \bigoplus_{k=0}^n \mathcal{H}_k \subseteq \bigoplus_{k=0}^n \widetilde{\mathcal{H}}_k \tag{17.22}$$

because the orthogonality proved above then gives $\widetilde{\mathcal{H}}_n = \mathcal{H}_n$ for all $n \geqslant 0$.

By Lemma 11.6 and $\mathcal{H}_1 = \mathrm{GHS}(B)$, (17.22) holds (with equality) for $n = 1$. For $n \geqslant 2$ we will prove (17.22) by induction. Suppose (17.22) holds up to and including index n. We then claim

$$\forall n \geqslant 0 \, \forall f \in L^2(D_n \cap [0,t]^n) \, \forall s \in [0,t]\colon \quad B_s I_t^{(n)}(f) \in \bigoplus_{k=0}^{n+1} \widetilde{\mathcal{H}}_k \tag{17.23}$$

Then, assuming (17.22) holds for n, the fact that $\bigoplus_{k=0}^n \mathcal{H}_k$ contains all products of the form $B_{s_1} \dots B_{s_k}$ for $0 \leqslant s_1, \dots, s_k \in [0,t]$ and $k = 1, \dots, n$ then implies that $\bigoplus_{k=0}^{n+1} \widetilde{\mathcal{H}}_k$ contains all such products for $k = 1, \dots, n+1$. Hence, $\bigoplus_{k=0}^{n+1} \mathcal{H}_k \subseteq \bigoplus_{k=0}^{n+1} \widetilde{\mathcal{H}}_k$ and the induction can proceed.

It remains to prove (17.23). By linearity and L^2-continuity of $f \mapsto I_t^{(n)}(f)$, it suffices to do this for f simple and, in fact, f of the form (17.12). Writing B_s as sum of the differences of the form $B_{b_k} - B_{a_k}$, $B_{a_{k+1}} - B_{b_k}$ and, possibly, $B_s - B_{a_k}$ or $B_s - B_{b_k}$, the quantity $B_s I_t^{(n)}(f)$ is then the sum of terms of the form $I_t^{(n+1)}(g)$, for g as in (17.12), plus terms of the form

$$(B_{b_k \wedge t} - B_{a_k \wedge t})^2 \prod_{\substack{i=1,\dots,n \\ i \neq k}} (B_{b_i \wedge t} - B_{a_i \wedge t}) \tag{17.24}$$

for some $k \in \{1, \dots, n\}$. Abbreviate $x_m(i) := a_k + (b_k - a_k)i2^{-m}$. Writing

$$\begin{aligned}(B_{b_k \wedge t} - B_{a_k \wedge t})^2 &= (b_k \wedge t - a_k \wedge t) \\ &\quad + \lim_{m\to\infty} \sum_{1 \leqslant i < j \leqslant 2^m} 2\bigl(B_{x_m(i)\wedge t} - B_{x_m(i-1)\wedge t}\bigr)\bigl(B_{x_m(j)\wedge t} - B_{x_m(j-1)\wedge t}\bigr)\end{aligned} \tag{17.25}$$

then shows that (17.24) is a limit of expressions of the form $I^{(n+1)}(g) + I^{(n-1)}(h)$ for g and h simple (of appropriate dimensionality), thus proving (17.23). □

The statement now gives different proofs of results we already established earlier using the "single-variable" stochastic calculus. Indeed, the following already appeared as Theorem 7.1:

Corollary 17.4 *Let $B = \{B_s : s \leqslant t\}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, P)$. Then for any $t \geqslant 0$ and any $X \in L^2(\Omega, \widetilde{\mathcal{F}}^B_t, P)$, where $\widetilde{\mathcal{F}}^B_t$ is as above, there is $Y \in \mathcal{V}_B$ so that*

$$X = EX + \int_0^t Y_s \, \mathrm{d}B_s \tag{17.26}$$

Proof. By Corollary 14.3 and Theorem 17.3, for any $X \in L^2(\Omega, \mathcal{F}^B, P)$ there are $\{f_n\}_{n\geqslant 0}$ with f_0 a constant and $f_n \in L^{2,\mathrm{loc}}(D_n)$ for all $n \geqslant 1$ such that

$$X = f_0 + \sum_{n\geqslant 1} I^{(n)}_t(f_n) \tag{17.27}$$

with the sum convergent in L^2. Taking expectations shows $f_0 = EX$ while Proposition 17.1 in turn guarantees that, for each $n \geqslant 1$, there is $Y^{(n)} \in \mathcal{V}$ such that

$$I^{(n)}_t(f_n) = \int_0^t Y^{(n)}_s \, \mathrm{d}B_s \tag{17.28}$$

Noting that $\|Y^{(n)}\|_{L^2([0,t]\otimes\Omega)} = \|I^{(n)}_t(f_n)\|_{L^2}$, the sum in

$$Y_s := \sum_{n\geqslant 1} Y^{(n)}_s \tag{17.29}$$

converges in $L^2([0,t] \otimes \Omega)$ and yields $Y \in \mathcal{V}$ so that (17.26) holds. □

Similarly, the following appeared as Theorem 7.2:

Corollary 17.5 (Itô representation theorem) *Let $B = \{B_t : t \geqslant 0\}$ be a standard Brownian motion and let $M = \{M_t : t \geqslant 0\}$ be a continuous L^2-martingale adapted to $\{\widetilde{\mathcal{F}}^B_t\}_{t\geqslant 0}$ for $\widetilde{\mathcal{F}}^B_t$ as above. Then there is $Y \in \mathcal{V}_B$ so that*

$$\forall t \geqslant 0\colon \quad M_t = M_0 + \int_0^t Y_s \, \mathrm{d}B_s \quad \text{a.s.} \tag{17.30}$$

Proof. Thanks to Corollary 17.4, for each $t \geqslant 0$ there is $Y^{(t)} \in \mathcal{V}$ so that

$$M_t = M_0 + \int_0^t Y^{(t)}_s \, \mathrm{d}B_s \quad \text{a.s.} \tag{17.31}$$

We thus need to show that the t-dependence of the integrand can be ruled out. To this end, the fact that the stochastic integral is a martingale then shows, for each $u \leqslant t$,

$$M_u = E(M_t|\mathcal{F}_u^B) = M_0 + \int_0^u Y_s^{(t)}\,\mathrm{d}B_s \quad \text{a.s.} \tag{17.32}$$

Comparing with (17.31) for $t := u$ gives

$$\int_0^u \bigl(Y_s^{(t)} - Y_s^{(u)}\bigr)\mathrm{d}B_s = 0 \tag{17.33}$$

thus implying that $Y^{(t)} = Y^{(u)}$ as elements of $L^2([0,u] \times \Omega)$. We now define

$$\forall n \geqslant 1\, \forall s \in [n-1, n)\colon \quad Y_s := Y_s^{(n)} \tag{17.34}$$

and note that $Y = Y^{(n)}$ as elements of $L^2([0,n] \times \Omega)$ by above reasoning. In particular, $Y \in \mathcal{V}$ and (17.30) holds, as desired. □

The connection between Itô and Wiener approaches are best exhibited in the celebrated formula:

Corollary 17.6 *Writing h_n for the n-th Hermit polynomial normalized so that $h_n(x) - x^n$ is a polynomial of degree less than n, we have*

$$\forall n \geqslant 1\colon \quad n! \int_0^1 \Bigl(\int_0^{t_n} \Bigl(\dots \Bigl(\int_0^{t_2} \mathrm{d}B_{t_1}\Bigr) \dots \Bigr)\mathrm{d}B_{t_{n-1}}\Bigr)\mathrm{d}B_{t_n} = h_n(B_1) \tag{17.35}$$

We leave a proof of this to the reader. We will see some application of the above expansions in the forthcoming lectures.