16. Wick ordered product

Here we continue the discussion of the Wick ordered product focusing on its action beyond just polynomials. Recall the concept of higher-order Paley-Wiener integrals with respect to a white noise that we introduced in Theorem 15.4. Our first observation is that the higher-order integrals naturally appear under Wick products:

Lemma 16.1 *For all $n \geqslant 1$ and all $f_1, \dots, f_n \in L^2(\mathscr{X}, \mathcal{G}, \mu)$, we have*

$$:\!\left(\prod_{k=1}^n \int f_k \mathrm{d}W\right)\!: \;= \int f_1 \otimes \cdots \otimes f_n \;\mathrm{d}\!:\!W^{\otimes n}\!: \tag{16.1}$$

where, for the purpose of this statement, $f_1 \otimes \cdots \otimes f_n \colon \mathscr{X}^n \to \mathbb{R}$ is defined by

$$f_1 \otimes \cdots \otimes f_n(x_1, \dots, x_n) := \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} \prod_{i=1}^n f_i(x_{\pi(i)}) \tag{16.2}$$

with $\mathcal{S}_n$ denoting the set of all permutations of $\{1, \dots, n\}$.

We leave the easy proof of this lemma to the reader while noting that this expression is sometimes used as a basis for a definition of the higher-order integrals. The appearance of the symmetrized tensor product shows again a connection with quantum bosons that are generally described by functions that are symmetric under any permutation of the coordinates. (Namely, $f_1 \otimes \cdots \otimes f_n$ describes the "state" in which one boson is in "state" f_1, another one in "state" f_2, etc.

The operation of taking Wick-ordered product of monomials leads to a natural product of elements in $L^2(\Omega, \mathcal{F}^W, P)$. This is defined as follows: Given any $m, n \in \mathbb{N}$ and $X \in \mathcal{H}_m$ and $Y \in \mathcal{H}_n$, set

$$X \odot Y := \mathrm{proj}_{\mathcal{H}_{n+m}}(XY) \tag{16.3}$$

In order to give some intuition about this, recall that each such X is a convergent sum of terms of the form

$$X_{t_1} \dots X_{t_m} + \text{lower order terms} \tag{16.4}$$

and similarly Y is a convergent sum of terms of the form

$$Y_{t_1} \dots Y_{t_n} + \text{lower order terms} \tag{16.5}$$

where the "lower order terms" are then such that the whole expression belongs to $\mathcal{H}_m$, resp., $\mathcal{H}_n$. It follows that XY is then a convergent sum of terms of the form

$$X_{t_1} \dots X_{t_m} Y_{t_1} \dots Y_{t_n} + \text{lower order terms} \tag{16.6}$$

but here the lower order terms no longer necessarily put the result into $\mathcal{H}_{m+n}$ and so we need to invoke the projection as well. All that the product does is to keep track of the leading order terms; the other terms just "ride along."

With $\odot$ defined between any pair of elements in $\bigcup_{n \in \mathbb{N}} \mathcal{H}_n$, we now extend the map linearly to turn it into an bilinear map $\odot$ of the linear vector space

$$\bigcup_{n \in \mathbb{N}} \mathrm{span}(\mathcal{H}_0 \cup \cdots \cup \mathcal{H}_n). \tag{16.7}$$

into itself. A natural question is then: Can this bilinear map be extended to all of $L^2(\Omega, \mathcal{F}^W, P)$? The answer to this is two-fold. When we restrict to elements from any of the linear subspaces $\text{span}(\mathcal{H}_0 \cup \cdots \cup \mathcal{H}_n)$, the extension does exists thanks to:

Lemma 16.2 *For all $m, n \in \mathbb{N}$ there exists $c_{m,n}$ such that*

$$\forall X \in \mathcal{H}_n \, \forall Y \in \mathcal{H}_m \colon \; \|X \odot Y\| \leqslant c_{m,n} \|X\| \|Y\| \tag{16.8}$$

We leave a proof of this to a homework exercise. Unfortunately, even the best coefficients $c_{m,n}$ one can choose above grow with m and n so that extension to $L^2(\Omega, \mathcal{F}^W, P)$ does not exist. We will demonstrate this by considering an function

$$f(x) := \sum_{n \in \mathbb{N}} a_n x^n \tag{16.9}$$

where we assume that the coefficients are such that $\sum_{n \in N} |a_n| r^n < \infty$ for each $r > 0$. For $X \in \mathcal{H}_1$ we then set

$$:f(X): \; := \sum_{n \in \mathbb{N}} a_n :X^n: \tag{16.10}$$

whenever the sum converges in L^2. To check conditions for that, note that then the Wick pairing formula gives

$$\begin{aligned} E\bigl([:f(X:]^2\bigr) &= \sum_{n \geqslant 0} |a_n|^2 E\bigl(:X^n: \, :X^n:\bigr) \\ &= \sum_{n \geqslant 0} |a_n|^2 n! \|X\|^{2n} \end{aligned} \tag{16.11}$$

This means that L^2-convergence in (16.10) generally requires that $\sum_{n \geqslant 0} |a_n|^2 n! r^n < \infty$ for each $r > 0$ which is stronger than $\sum_{n \in N} |a_n| r^n < \infty$ needed for f to be defined by the series (16.9). (E.g., $a_n := 1/\sqrt{n!}$ would work for (16.9) but not (16.11).)

Some important functions are still accessible, for instance:

Lemma 16.3 *For each $X \in \mathcal{H}_1$, $:\mathrm{e}^X:$ is well defined and, in fact,*

$$:\mathrm{e}^X: \; = \mathrm{e}^{X - \frac{1}{2}\|X\|^2} \tag{16.12}$$

Proof. Suppose without loss of generality that $\|X\| > 0$. The usual series representation of the exponential along with the fact that, as shown in Lemma 14.4, $:Z^n: \; = h_n(Z)$ for each $Z = \mathcal{N}(0,1)$ then give

$$\begin{aligned} :\mathrm{e}^X: \; &= \sum_{n \geqslant 0} \frac{1}{n!} :X^n: \; = \sum_{n \geqslant 0} \frac{1}{n!} \|X\|^n h_n\Bigl(\frac{X}{\|X\|}\Bigr) \\ &= \sum_{n \geqslant 0} \frac{1}{n!} \|X\|^n \; \mathrm{e}^{t^2/2} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \mathrm{e}^{-t^2/2} \Big|_{t := \frac{X}{\|X\|}} \\ &= \mathrm{e}^{t^2/2} \mathrm{e}^{-(t - \|X\|)^2/2} \Big|_{t := \frac{X}{\|X\|}} \\ &= \exp\Bigl\{\frac{1}{2}\Bigl(\frac{X}{\|X\|}\Bigr)^2 - \frac{1}{2}\Bigl(\frac{X}{\|X\|} - \|X\|\Bigr)^2\Bigr\} \end{aligned} \tag{16.13}$$

where we first invoked the definition of h_n and then interpreted the result using the Taylor series whose convergence is justified by the analyticity of the exponential. A simple computation now shows that the right-hand side equals $\mathrm{e}^{X-\frac{1}{2}\|X\|^2}$ as desired. $\square$

Note that the above shows that $:\mathrm{e}^X: \; > 0$ even though the Wick product does not generally preserve positivity (e.g., $:X^2: \; = X^2 - \|X\|^2$ which can be of both signs.) Note also that the combination of Lemmas 16.1 and 16.3 gives:

Corollary 16.4 *Let $f \in L^2(\mathscr{X}, \mathcal{G}, \mu)$. Then*

$$:\mathrm{e}^{\int f \mathrm{d}W}: \; = 1 + \sum_{n\geqslant 1} \frac{1}{n!} \int \underbrace{f \otimes \cdots \otimes f}_{n\text{-times}} \mathrm{d}:W^{\otimes n}: \tag{16.14}$$

We remark that this representation is formally similar to the so called *time-ordered exponential* which is perturbative way of writing a solution to the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} Z(t) = A(t) Z(t) \tag{16.15}$$

for vector-valued variable $Z(t)$ with initial value $Z(0)$ as follows

$$Z(t) = Z(0) + \sum_{n\geqslant 1} \int_{0<t_1<\cdots<t_n\leqslant t} A(t_n) \dots A(t_1) Z(0) \, \mathrm{d}t_1 \dots \mathrm{d}t_n \tag{16.16}$$

Here there is no $n!$ term because we are only integrating over $1/n!$-portion of the multiinterval $[0,t]^n$. The implicit presence of the $1/n!$ means that, assuming $t \mapsto A(t)$ not growing too fast, the infinite series converges along with its derivative with respect to t. We will come back to this one more time once we have introduced iterated Itô integrals, where the analogy is even more striking.