15. Higher-order chaos integrals

Here we will continue the development of the integrals with respect to higher-order chaos for Wiener chaos expansion of a white noise. We will later show how this specializes to the case of iterated Itô integrals.

15.1 Wick ordered product.

We will start with a concept that was already alluded to in the previous lecture. Consider a mean-zero Gaussian process $\{X_t : t \in T\}$, for which we recall $\mathcal{H}_1 = \text{GHS}(X)$. For each $n \geqslant 1$ and each $Y_1 \dots, Y_n \in \text{GHS}(X)$, define the symbol

$$:Y_1 \dots Y_n: \ := \text{proj}_{\mathcal{H}_n}(Y_1 \dots Y_n). \tag{15.1}$$

where $\mathcal{H}_n$ is the n-th order Wiener chaos space associated with process X. We call $:Y_1 \dots Y_n:$ the Wick ordered product, or normal-ordered product, of $Y_1 \dots Y_n$.

We will now discuss some basic properties of this object as that is necessary for our later derivations in the context of the white noise. Throughout, we will need the following technical tool, bearing also Wick's name:

Lemma 15.1 (Wick pairing) *Let $Z_1, \dots, Z_n$ be centered multivariate normal and Q a polynomial in n variables. Writing $\partial_k Q$ for the derivative of Q in the k-th variable,*

$$E\big(Z_1 Q(Z_1, \dots, Z_n)\big) = \sum_{i=1}^{n} E(Z_1 Z_k) E\Big(\frac{\partial Q}{\partial x_k}(Z_1, \dots, Z_n)\Big) \tag{15.2}$$

In particular, for n even,

$$E\bigg(\prod_{i=1}^{n} Z_i\bigg) = \sum_{\pi \in \mathcal{P}_n} \prod_{(i,j)\in\pi} E(Z_i Z_j) \tag{15.3}$$

where $\mathcal{P}_n$ is the set of pairings of indices in $\{1, \dots, n\}$; i.e., partitions of $\{1, \dots, n\}$ into sets of size two that are then written as ordered pairs.

Proof. The first identity is checked readily using Gaussian integration by parts. The second identity is proved by induction from the first. □

Using the above, for all $Y_1, Y_2, Y_3 \in \mathcal{H}_1$ we now readily check that

$$\begin{aligned} :Y_1: \ &= Y_1 \\ :Y_1 Y_2: \ &= Y_1 Y_2 - \text{Cov}(Y_1 Y_2) \end{aligned} \tag{15.4}$$

and

$$:Y_1 Y_2 Y_3: \ = Y_1 Y_2 Y_3 - \text{Cov}(Y_1 Y_2) Y_3 - \text{Cov}(Y_1 Y_3) Y_2 - \text{Cov}(Y_2 Y_3) Y_1 \tag{15.5}$$

(For instance, (15.5) requires checking that the expectation $E(U :Y_1 Y_2 Y_3:)$ vanishes for U a constant, $U = Z$ and $U = XT$, for any $Z, T \in \mathcal{H}_1$.) Since, for W a white noise on $(\mathscr{X}, \mathcal{G}, \mu)$, the above gives

$$:W(A)W(B): \ = W(A)W(B) - \mu(A \cap B) \tag{15.6}$$

the second line above justifies our earlier use of the double colon notation.

The expression on the right of (15.4–15.5) are composed of terms of the same parity as is the product, and that the leading order term is the product itself. This is not an accident, as the next lemma shows:

Lemma 15.2 *For all $n \geqslant 1$ and all $Y_1, \dots, Y_n \in \mathcal{H}_1$,*

$$:Y_1 \dots Y_n: \; = \sum_{\substack{A \subseteq \{1,\dots,n\} \\ |A| \text{ even}}} (-1)^{|A|/2} \Biggl[\sum_{\pi \in \mathcal{P}(A)} \prod_{(i,j) \in \pi} E(Y_i Y_j) \Biggr] \prod_{i \notin A} Y_i \tag{15.7}$$

where $\mathcal{P}(A)$ is the set of all pairings from A; namely, the partitions of A into two-point sets that are written as ordered pairs.

Proof. Write $p(Y_1, \dots, Y_n)$ for the right-hand side. Noting that $p(Y_1, \dots, Y_n) - Y_1 \dots Y_n$ belongs to the linear span of $\bigcup_{k<n} \mathcal{H}_k$, it suffices to show that $Ep(Y_1, \dots, Y_n) = 0$ and, for any $k < n$ and any $Z_1, \dots, Z_k \in \mathcal{H}_1$, also $E(Z_1 \dots Z_k p(Y_1, \dots, Y_n)) = 0$. (Indeed, this implies that the projection of $Y_1 \dots Y_n$ and $p(Y_1, \dots, Y_n)$ on $\mathcal{H}_n$ are equal.)

We will prove these together. Consider the expectation $E(Z_1 \dots Z_k p(Y_1, \dots, Y_n)) = 0$ where, for $k = 0$, the Z-terms multiplying $p(Y_1, \dots, Y_n)$ are absent. Writing out the explicit form of $p(Y_1, \dots, Y_n)$ using the Wick pairing lemma, we get

$$\begin{aligned} &E(Z_1 \dots Z_k p(Y_1, \dots, Y_n)) \\ &\qquad = \sum_{\substack{A \subseteq [n] \\ |A| \text{ even}}} (-1)^{|A|/2} \Biggl[\sum_{\pi \in \mathcal{P}(A)} \prod_{(i,j) \in \pi} E(Y_i Y_j) \Biggr] E\Biggl(\Bigl(\prod_{i \in A^{\mathrm{c}}} Y_i \Bigr) \Bigl(\prod_{j=1}^{k} Z_j \Bigr) \Biggr). \end{aligned} \tag{15.8}$$

where $[n] := \{1, \dots, n\}$ and $A^{\mathrm{c}} := [n] \smallsetminus A$. The term in the square brackets is to be interpreted as 1 when $A = \varnothing$.

The Wick pairing formula tells us how to compute the last expectation in (15.8) using pairings. In any pairing, some Y's in A^{c} get paired together and others get paired with the Z's. Denoting by B the set of Y's that are ultimately paired with each other, the pairing of A can then be any subset of the pairing of B. Noting that each pairing of A has exactly $|A|/2$ elements, this rewrites (15.8) as

$$\begin{aligned} &E\bigl(Z_1 \dots Z_k p(Y_1, \dots, Y_n)\bigr) \\ &\qquad = \sum_{\substack{B \subseteq [n] \\ |B| \text{ even} \\ n-|B| \leqslant k}} \Biggl[\sum_{\pi \in \mathcal{P}(B)} \sum_{\pi' \subseteq \pi} (-1)^{|\pi'|} \prod_{(i,j) \in \pi} E(Y_i Y_j) \Biggr] \Xi(B, k) \end{aligned} \tag{15.9}$$

where for $k = 0$ we set $\Xi(B, 0) := 1$ while for $k \neq 0$ we let

$$\Xi(B,k) := \sum_{\substack{C \subseteq [k] \\ |C| = n-|B|}} \sum_{\substack{\sigma\colon C \to B^{\mathrm{c}} \\ \text{bijection}}} \prod_{i \in C} E(Z_i, Y_{\sigma(i)}) \Biggl[\sum_{\pi \in \mathcal{P}([n] \smallsetminus C)} \prod_{(i,j) \in \pi} E(Z_i Z_j) \Biggr] \tag{15.10}$$

Here the term in the square brackets equals one if $C = [n]$. Above C is the set of indices in $[k]$ of Z's that get paired with Y's. The remaining Z's are paired up with other Z's.

We now recall the identity underlying the proof of the Möbius inversion formula:

$$\sum_{\pi'\subseteq\pi}(-1)^{|\pi'|}=\bigl(1+(-1)\bigr)^{|\pi|} \tag{15.11}$$

As $|\pi|=|B|/2$ this shows that the square bracket in (15.9) equals zero unless $B=\varnothing$ which means that $|C|=n$ in (15.13). But that is impossible as $C\subseteq[k]$ and $k<n$. Hence $E(Z_1\dots Z_k p(Y_1,\dots,Y_n))=0$ for all $k=0,\dots,n-1$, proving the claim. □

The previous proof has one additional important consequence:

Lemma 15.3 *For all $n\geqslant 1$ and all $Y_1,\dots,Y_n,Z_1,\dots,Z_n\in\mathcal{H}_1$,*

$$E\bigl(:Y_1\dots Y_n:\ :Z_1\dots Z_n:\bigr)=\sum_{\pi\in\mathcal{S}_n}\prod_{i=1}^n E(Y_iZ_{\pi(i)}), \tag{15.12}$$

where $\mathcal{S}_n$ is the set of all permutations of $\{1,\dots,n\}$.

Proof. By the fact that $:Y_1\dots Y_n:$ and $:Z_1\dots Z_n:$ are both orthogonal projections on $\mathcal{H}_n$, the expectation equals $E(:Y_1\dots Y_n:\ Z_1\dots Z_n)$. The calculation in the previous proof then equates this with $\Xi(\varnothing,n)$ in which $C=[n]$ and so the bracket term on the right of (15.10) is one. Representing bijections as permutations gives the claim. □

Note that the above can be summarized by saying that, under expectation, the terms in a Wick-ordered product are not paired up with one another when the Wick pairing formula is invoked.

15.2 Higher order Paley-Wiener integrals.

Let us go back to the discussion we had in the previous lecture, but now done for general $\mathcal{H}_n$. Using the above notation

$$\mathcal{H}_n=\overline{\operatorname{span}\Bigl\{:W(A_1)\dots W(A_n):\,:\,A_1,\dots,A_n\in\mathcal{G}\Bigr\}}^{L^2(\Omega,\mathcal{F}^W,P)} \tag{15.13}$$

Every term in the linear span takes the form

$$\int f\,\mathrm{d}{:}W^{\otimes n}{:}\ :=\sum_{i_1,\dots,i_n=1}^m a_{i_1,\dots,i_n}:W(A_{i_1})\dots W(A_{i_n}): \tag{15.14}$$

for some simple function $f\colon\mathscr{X}^n\to\mathbb{R}$ of the form

$$f=\sum_{i_1,\dots,i_n=1}^m a_{i_1,\dots,i_n}1_{A_{i_1}\times\dots\times A_{i_n}} \tag{15.15}$$

Thanks to the additivity of $A_1,\dots,A_n\mapsto :W(A_{i_1})\dots W(A_{i_n}):$, the expression on the right of (15.14) does not depend on the representation of f. For reasons explained earlier, we will restrict f to functions that are symmetric under permutations of the indices, i.e., such that

$$\forall\pi\in\mathcal{S}_n\,\forall x_1,\dots,x_n\in\mathscr{X}:\ f(x_{\pi(1)},\dots,x_{\pi(n)})=f(x_1,\dots,x_n) \tag{15.16}$$

The restriction to such functions does not have an effect on the integral, as the antisymmetric part drops out anyway.

We will write $L^2_{\text{sym}}(\mathscr{X}^n,\mathcal{G}^{\otimes n},\mu^{\otimes n})$ for the closed linear subspace of $L^2_{\text{sym}}(\mathscr{X}^n,\mathcal{G}^{\otimes n},\mu^{\otimes n})$ consisting of functions satisfying (15.16). We are now ready for:

Theorem 15.4 *Let $n\geqslant 1$. For each simple $f\colon \mathscr{X}^n\to\mathbb{R}$ satisfying* (15.16)*,*

$$E\biggl(\Bigl(\int f\ \mathrm{d}{:}W^{\otimes n}{:}\ \Bigr)^2\biggr)=n!\int f^2\mathrm{d}\mu^{\otimes n}\tag{15.17}$$

In particular, $f\mapsto\int f\ \mathrm{d}{:}W^{\otimes n}{:}$ extends uniquely to a continuous linear map

$$L^2_{\text{sym}}(\mathscr{X}^n,\mathcal{G}^{\otimes n},\mu^{\otimes n})\to L^2(\Omega,\mathcal{F}^W,P)\tag{15.18}$$

that is an isometry modulo the prefactor $n!$ in (15.17)*. In particular, we have*

$$\mathcal{H}_n=\biggl\{\int f\ \mathrm{d}{:}W^{\otimes n}{:}\ \colon f\in L^2_{\text{sym}}(\mathscr{X}^n,\mathcal{G}^{\otimes n},\mu^{\otimes n})\biggr\}\tag{15.19}$$

and for each $Y\in L^2(\Omega,\mathcal{F}^W,P)$ and $n\geqslant 1$ there exists $f_n\in L^2_{\text{sym}}(\mathscr{X}^n,\mathcal{G}^{\otimes n},\mu^{\otimes n})$ such that

$$Y=EY+\sum_{n=1}^\infty\int f_n\ \mathrm{d}{:}W^{\otimes n}{:}\tag{15.20}$$

where the sum converges in L^2.

Proof. Write f in the form (15.15) where $A_1,\dots,A_n$ are without loss of generality disjoint and $(i_1,\dots,i_n)\mapsto a_{i_1,\dots,i_n}$ is symmetric under permutations. Lemma 15.3 then gives

$$\begin{aligned}
&E\biggl(\Bigl(\int f\ \mathrm{d}{:}W^{\otimes n}{:}\ \Bigr)^2\biggr)\\
&\quad=\sum_{i_1,\dots,i_n}\sum_{j_1,\dots,j_n}a_{i_1,\dots,i_n}a_{j_1,\dots,j_n}E\Bigl({:}W(A_{i_1})\dots W(A_{i_n}){:}\ {:}W(A_{j_1})\dots W(A_{j_n}){:}\ \Bigr)\\
&\quad=\sum_{i_1,\dots,i_n}\sum_{j_1,\dots,j_n}\sum_{\pi\in\mathcal{S}_n}a_{i_1,\dots,i_n}a_{j_1,\dots,j_n}\prod_{i=1}^n\mu\bigl(A_{i_k}\cap A_{j_{\pi(k)}}\bigr)\\
&\quad=\sum_{i_1,\dots,i_n}\sum_{j_1,\dots,j_n}\sum_{\pi\in\mathcal{S}_n}a_{i_1,\dots,i_n}a_{j_{\pi(1)},\dots,j_{\pi(n)}}\prod_{i=1}^n\mu\bigl(A_{i_k}\cap A_{j_k}\bigr)
\end{aligned}\tag{15.21}$$

where in the last step we first wrote π^{-1} instead of π and then transferred the permutation to the indices $j_1,\dots,j_n$. The disjointness forces $\mu(A_{i_k}\cap A_{j_k})=\delta_{i_k,j_k}\mu(A_{i_k})$ which using that the coefficients are permutation symmetric yields

$$E\biggl(\Bigl(\int f\ \mathrm{d}{:}W^{\otimes n}{:}\ \Bigr)^2\biggr)=\sum_{\pi\in\mathcal{S}_n}\sum_{i_1,\dots,i_n}(a_{i_1,\dots,i_n})^2\prod_{i=1}^n\mu(A_{i_1})\tag{15.22}$$

The sum over permutation results in the factor $n!$; the rest of the expression is then identified with the integral of f^2 against $\mu^{\otimes n}$.

The isometry (up to $n!$) proves existence of the unique extension to a map (15.18). Using (15.13) we then get (15.19). This with the help of Corollary 14.3 gives (15.20). $\square$

We remark that the above gives $L^2(\Omega, \mathcal{F}^W, P)$ the representation as the so called *Fock space*. Indeed, $f \in L^2_{\mathrm{sym}}(\mathscr{X}^n, \mathcal{G}^{\otimes n}\mu^{\otimes n})$ can be thought of as a test function in n-variables of particles that are indistinguishable, which is what is represented by the permutation symmetry. (In quantum mechanics, permutations can still produce complex-modulus one multiples of f but do not for particles that are called bosons.) The subspace $\mathcal{H}_n$ is thus a subspace corresponding to states with n bosons.