13. Cameron-Martin theorem

We will now use the structure of the Gaussian Hilbert space to generalize Theorem 10.4 that dealt with the special case of standard Brownian motion.

13.1 The statement and easy direction of proof.

A natural way to generalize the question asked and answered by Cameron and Martin for the case of the standard Brownian motion is as follows:

> Given a sample of a Gaussian process $\{X_t \colon t \in T\}$, for what functions $f \colon T \to \mathbb{R}$ is its "shift by f" — meaning, the sample $\{X_t + f(t) \colon t \in T\}$ — indistinguishable from X?

As we have only one sample to compare to, we posit that this occurs exactly when the laws of the two processes are mutually absolutely continuous with respect to each other. A necessary and sufficient conditions for exactly that comes in:

Theorem 13.1 (Cameron-Martin Theorem, general form) *Let $\{X_t \colon t \in T\}$ be a mean-zero Gaussian process. Then, with the laws taken on the product space $(\mathbb{R}^T, \mathcal{B}(\mathbb{R})^{\otimes T})$,*

$$\forall f \in \mathbb{R}^T\colon \quad \text{Law}(X+f) \ll \text{Law}(X) \;\Leftrightarrow\; f \in \text{CMS}(X). \tag{13.1}$$

Moreover, for all $f \in \text{CMS}(X)$,

$$\forall A \in \mathcal{B}(\mathbb{R})^{\otimes T}\colon \quad P(X+f \in A) = E\big(1_A(X)\,\mathrm{e}^{Y-\frac12 E(Y^2)}\big), \tag{13.2}$$

where $Y = \phi^{-1}(f)$. The absolute continuity "$\ll$" in (13.1) *can be replaced by equivalence "$\sim$."*

A few remarks are in order. First, the restriction to mean-zero processes is not merely a convenience; indeed, the result implies:

Corollary 13.2 *Let $\{X_t \colon t \in T\}$ be a Gaussian process, not necessarily mean zero. Then* $\text{Law}(X - EX)$ *and* $\text{Law}(X)$ *are mutually absolutely continuous if and only if $t \mapsto E(X_t)$ lies in* $\text{CMS}(X)$, *and they are mutually singular otherwise.*

Second, note also that we are not assuming any kind of regularity of X. This forces us to work with laws interpreted in the coarsest possible way; namely, on the product space $(\mathbb{R}^T, \mathcal{B}(\mathbb{R})^{\otimes T})$. The advantage of that is that one only needs the finite-dimensional distributions to determine the whole law. This is enough to compare the laws of continuous processes over separable metric spaces as there the projection of the process on any countable dense subsets determines the law.

As in Theorem 10.4, the proof of one direction of the implication is easy and the other one is considerably more laborious. The easy direction comes in:

Proof of Theorem 13.1, "$\Leftarrow$" in (13.1). Let $f \in \text{CMS}(X)$ and $Y := \phi^{-1}(f)$. Set

$$\widetilde{P}(A) := E\big(1_A\,\mathrm{e}^{Y-\frac12 E(Y^2)}\big) \tag{13.3}$$

and write $\widetilde{E}$ for expectation with respect to $\widetilde{P}$. Then, repeating the argument form the proof of Girsanov's Theorem, for any $t_1, \dots, t_n \in R$, $\lambda_1, \dots, \lambda_n \in \mathbb{R}$,

$$\begin{aligned}\widetilde{E}\biggl(\exp\Bigl\{\sum_{j=1}^n \lambda_j\bigl(X_{t_j} - f(t_j)\bigr) - \frac{1}{2}\text{Var}\Bigl(\sum_{j=1}^n \lambda_j X_{t_j}\Bigr)\Bigr\}\biggr)\\ = E\biggl(\exp\Bigl\{\Bigl(Y + \sum_{j=1}^n \lambda_j X_{t_j}\Bigr) - \frac{1}{2}\text{Var}\Bigl(Y + \sum_{j=1}^n \lambda_j X_{t_j}\Bigr)\Bigr\}\biggr) = 1\end{aligned} \tag{13.4}$$

where we used that $\text{Cov}(Y, X_{t_j}) = f(t_j)$ to complete the square. Since the finite-dimensional distributions determine the law on $\mathbb{R}^T$, it follows that

$$\text{Law}(X - f) \text{ under } \widetilde{P} \;=\; \text{Law}(X) \text{ under } P \tag{13.5}$$

thus proving (13.2) via a deterministic shift, and the direction "$\Leftarrow$" in (13.1), by the fact that $\widetilde{P} \ll P$. (In fact, we get $P \ll \widetilde{P}$ as well, since the exponential is positive a.s.) $\square$

13.2 Proof of the harder direction.

We will now move to the opposite direction. In the proof, we will make use of the fact, stated and proved as Lemma 8.2 earlier, that the result is for finite T proved easily with the help of linear algebra. For the extension to infinite T, we need first to set up the necessary notation.

Assume that X is realized on a probability space $(\Omega, \mathcal{F}, P)$. For each $S \subseteq T$, denote

$$\mathcal{F}_S := \sigma(X_t \colon t \in S). \tag{13.6}$$

We start by processing the assumption that the law of $X + f$ is absolutely continuous with respect to that of X:

Lemma 13.3 *Let $f \colon T \to \mathbb{R}$ be such that $\text{Law}(X + f) \ll \text{Law}(X)$ on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R})^{\otimes T})$. Then there exists $M \in L^1(\Omega, \mathcal{F}_T, P)$ such that*

$$M \geqslant 0 \;\wedge\; \forall A \in \mathcal{B}(\mathbb{R})^{\otimes T}\colon\; P(X + f \in A) = E\bigl(1_A(X)M\bigr). \tag{13.7}$$

Proof. Write $\mu(A) := P(X + f \in A)$, resp., $\nu(A) := P(X \in A)$ for the distributions of $X + f$, resp., X on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R})^{\otimes T})$. The assumption $\mu \ll \nu$ implies the existence of the Radon-Nikodym derivative $F \in L^1(\mathbb{R}^T, \mathcal{B}(\mathbb{R})^{\otimes T}, \nu)$ such that $F \geqslant 0$ and

$$\forall A \in \mathcal{B}(\mathbb{R})^{\otimes T}\colon\; \mu(A) = \int_A F \,\mathrm{d}\nu. \tag{13.8}$$

Denote $M := F \circ X$ and note that M is a random variable on $(\Omega, \mathcal{F}, P)$. Since each $A \in \mathcal{F}_T$ takes the form $A = \{X \in B\}$ for some $B \in \mathcal{B}(\mathbb{R})^{\otimes T}$, we then get $M \in L^1(\Omega, \mathcal{F}_T, P)$ as well as (13.7) by the standard change-of-variables formula for measures and integrals. $\square$

Recall that, despite using the term Hilbert space for $\text{GHS}(X)$, we will henceforth regard that as a space of random variables rather than equivalence classes thereof. With that in mind, consider the collection of sets

$$\mathcal{L} := \Bigl\{S \subseteq T\colon \Bigl(\exists Y_S \in \mathrm{GHS}(X)\colon \mathcal{F}_S\text{-measurable} \wedge E(M|\mathcal{F}_S) = \mathrm{e}^{Y_S - \frac{1}{2}E(Y_S^2)} \text{ a.s.}\Bigr)\Bigr\} \tag{13.9}$$

Note that Y_S is, if it exists, determined uniquely up to modifications on a $\mathcal{F}_S$-measurable P-null set. Our ultimate goal is to show that $T \in \mathcal{L}$ but that will have to be proved gradually. We start from:

Lemma 13.4 $\forall S \subseteq T\colon\ S$ finite $\Rightarrow\ S \in \mathcal{L}$

Proof. Abbreviate $\widetilde{P}(A) := E(1_A M)$. Let $S \subseteq T$ be finite and denote by C the covariance of $\{X_t\colon t \in S\}$. The restriction of $\widetilde{P}$ to S is absolutely continuous with respect to the same restriction of P and so, by Lemma 8.2,

$$\frac{\mathrm{d}\widetilde{P}|_{\mathcal{F}_S}}{\mathrm{d}P|_{\mathcal{F}_S}} = \exp\Bigl\{\sum_{t\in S} h(t)X_t - \frac{1}{2}\sum_{t,t'\in S} h(t)h(t')C(t,t')\Bigr\} \tag{13.10}$$

for $h := C^{-1}f$ where the existence of the right-hand side is part of the conclusion of Lemma 8.2. Denoting $Y_S := \sum_{t\in S} h(t)X_t$, we then have

$$\forall A \in \mathcal{F}_S\colon \quad \widetilde{P}(A) = E\bigl(1_A \mathrm{e}^{Y_S - \frac{1}{2}E(Y_S^2)}\bigr) \tag{13.11}$$

In light of uniqueness of the conditional expectation, this gives $E(M|\mathcal{F}_S) = \mathrm{e}^{Y_S - \frac{1}{2}E(Y_S^2)}$ a.s. Since Y_S is also $\mathcal{F}_S$-measurable, we have $S \in \mathcal{L}$ as claimed. □

We will naturally aim to approximate infinite S by sequences of finite sets. For that we need to compare Y_S and $Y_{\widetilde{S}}$ for sets such that $S \subseteq \widetilde{S}$:

Lemma 13.5 *For all $S, \widetilde{S} \in \mathcal{L}$,*

$$S \subseteq \widetilde{S} \quad \Rightarrow \quad (Y_{\widetilde{S}} - Y_S) \perp\!\!\!\perp Y_S \ \wedge\ E(Y_S^2) \leqslant E(Y_{\widetilde{S}}^2) \tag{13.12}$$

Proof. The explicit form of $E(M|\mathcal{F}_S)$ whenever $S \in \mathcal{L}$ along with the "smaller always wins" principle for conditional expectation give

$$\begin{aligned} E(M|\mathcal{F}_S) &= E\bigl(E(M|\mathcal{F}_{\widetilde{S}}) \,\big|\, \mathcal{F}_S\bigr) \\ &= E\Bigl(E(M|\mathcal{F}_S)\mathrm{e}^{Y_{\widetilde{S}} - Y_S - \frac{1}{2}[E(Y_{\widetilde{S}}^2) - E(Y_S^2)]} \,\Big|\, \mathcal{F}_S\Bigr) \\ &= E(M|\mathcal{F}_S)E\Bigl(\mathrm{e}^{Y_{\widetilde{S}} - Y_S - \frac{1}{2}[E(Y_{\widetilde{S}}^2) - E(Y_S^2)]} \,\Big|\, \mathcal{F}_S\Bigr) \end{aligned} \tag{13.13}$$

where we used that $E(M|\mathcal{F}_S)$ is $\mathcal{F}_S$-measurable to take it out of the last conditional expectation. Since $E(M|\mathcal{F}_S) > 0$ by $S \in \mathcal{L}$, we can cancel $E(M|\mathcal{F}_S)$ on both sides. Taking expectation then shows

$$1 = E\Bigl(\mathrm{e}^{Y_{\widetilde{S}} - Y_S - \frac{1}{2}[E(Y_{\widetilde{S}}^2) - E(Y_S^2)]}\Bigr) = \mathrm{e}^{\frac{1}{2}E((Y_S - Y_{\widetilde{S}})^2) - \frac{1}{2}[E(Y_{\widetilde{S}}^2) - E(Y_S^2)]} \tag{13.14}$$

which then gives

$$E\bigl((Y_S - Y_{\widetilde{S}})^2\bigr) - \frac{1}{2}\bigl[E(Y_{\widetilde{S}}^2) - E(Y_S^2)\bigr] = 0 \tag{13.15}$$

Cancelling repeated terms and rearranging the rest, this becomes

$$E\big(Y_S(Y_S - Y_{\widetilde{S}})\big) = 0 \tag{13.16}$$

thus implying $(Y_S - Y_{\widetilde{S}}) \perp Y_S$ and, by the fact that non-correlation implies independence for Gaussian random variables, $(Y_S - Y_{\widetilde{S}}) \perp\!\!\!\perp Y_S$. The inequality for second moments then follows readily from this. □

Our next aim is to see what happens when we take a limit:

Lemma 13.6 *For all $\{S_n\}_{n\in\mathbb{N}} \in \mathcal{L}^{\mathbb{N}}$ and all $S \subseteq T$,*

$$S_n \uparrow S \quad \Rightarrow \quad \sup_{n\geqslant 1} E(Y_{S_n}^2) < \infty \ \wedge \ S \in \mathcal{L} \ \wedge \ Y_{S_n} \to Y_S \text{ in } L^2 \tag{13.17}$$

Proof. Assume $S_n \uparrow S$. Since $\mathcal{F}_S = \sigma(\bigcup_{n\geqslant 1} \mathcal{F}_{S_n})$, the Lévy Forward Theorem then gives

$$E(M|\mathcal{F}_{S_n}) \underset{n\to\infty}{\longrightarrow} E(M|\mathcal{F}_S) \quad \text{a.s.} \tag{13.18}$$

If we had $E(Y_{S_n}^2) \to \infty$, then Chebyshev's inequality would give

$$P\big(Y_{S_n} > \tfrac{1}{4}E(Y_{S_n}^2)\big) \leqslant \frac{16}{E(Y_{S_n}^2)} \underset{n\to\infty}{\longrightarrow} 0 \tag{13.19}$$

The explicit form $E(M|\mathcal{F}_{S_n}) = \exp\{Y_{S_n} - E(Y_{S_n}^2)\}$ would then imply $E(M|\mathcal{F}_{S_n}) \to 0$ in probability. But that would force $E(M|\mathcal{F}_S) = 0$ a.s. via (13.18), in contradiction with $E(E(M|\mathcal{F}_S)) = EM = 1$. Hence, we must have $\sup_{n\geqslant 1} E(Y_{S_n}^2) < \infty$.

Lemma 13.5 shows that $\{Y_{S_{n+1}} - Y_{S_n}\}_{n\geqslant 0}$, with $Y_{S_0} := 0$, are independent centered with

$$\sum_{n\geqslant 1} E\big((Y_{S_{n+1}} - Y_{S_n})^2\big) = \sup_{n\geqslant 1} E(Y_{S_n}^2) < \infty \tag{13.20}$$

By the Kolmogorov inequality, there exists $\Omega^\star \in \mathcal{F}_S$ with $P(\Omega^\star) = 1$ such that

$$Y_{S_n} = Y_{S_1} + \sum_{k=1}^{n} (Y_{S_{k+1}} - Y_{S_k}) \underset{n\to\infty}{\longrightarrow} Y_S := Y_{S_1} + \sum_{n=1}^{\infty} (Y_{S_{n+1}} - Y_{S_n}) \tag{13.21}$$

on $\Omega^\star$. Setting $Y_S := 0$ on $\Omega \smallsetminus \Omega^\star$ yields an $\mathcal{F}_S$-measurable random variable such that $Y_{S_n} \to Y_S$ a.s. Thanks to the Gaussian nature of these random variables, the convergence is also in L^2 and $E(Y_S^2) = \lim_{n\to\infty} E(Y_{S_n}^2)$. As $Y_S \in \text{GHS}(X)$ and

$$E(M|\mathcal{F}_S) \overset{\text{a.s.}}{=} \lim_{n\to\infty} e^{Y_{S_n} - \frac{1}{2}E(Y_{S_n}^2)} = e^{Y_S - \frac{1}{2}E(Y_S^2)} \tag{13.22}$$

we conclude $S \in \mathcal{L}$ as desired. □

We record another simple consequence of the above:

Corollary 13.7 *For all $S' \subseteq T$ countable, we have $E(Y_{S'}^2) < \infty$ and $S' \in \mathcal{L}$. Moreover, there exists $S \subseteq T$ such that*

$$S \text{ countable} \ \wedge \ E(Y_S^2) = \sup\big\{E(Y_{S'}^2) \colon S' \in \mathcal{L} \text{ countable}\big\} \tag{13.23}$$

where the supremum is thus finite.

Proof. That $E(Y_{S'}^2) < \infty$ and $S' \in \mathcal{L}$ for $S' \subseteq T$ is proved by combining Lemma 13.4 with Lemma 13.6 and an approximation of S' by an increasing sequence of finite sets. For the second part of the claim, let $\{S_n\}_{n\in\mathbb{N}}$ be non-decreasing and such that $E(Y_{S_n}^2)$ tends to the supremum, whether finite or not. Lemma 13.6 allows us to find finite sets $S_n' \subseteq S_n$, with $\{S_n'\}_{n\in\mathbb{N}}$ still increasing, such that $E(Y_{S_n}^2) \leqslant E(Y_{S_n'}) + 2^{-n}$. But then $S_n \uparrow S := \bigcup_{n\in\mathbb{N}} S_n$ and $S_n' \uparrow S$, proving that $S \in \mathcal{L}$ and $E(Y_S^2) < \infty$. Clearly, $E(Y_S^2)$ equals the supremum, which is then necessarily finite. □

We are now ready to give:

Proof of Theorem 13.1, "⇒" in (13.1). Corollary 13.7 gives us existence of a countable set $S \subseteq T$ such that

$$S \in \mathcal{L} \;\wedge\; E(Y_S^2) = \sup\big\{E(Y_{S'}^2) \colon S' \in \mathcal{L} \text{ countable}\big\} \tag{13.24}$$

We claim that Y_S serves as Y_T, and thus $\mathcal{L}$ contains all subsets of T. Indeed, let $A \in \sigma(X_t \colon t \in T)$. By the "countability curse" there exists $\widetilde{S} \subseteq T$ countable such that $A \in \mathcal{F}_{\widetilde{S}}$. Without loss of generality we may assume that $S \subseteq \widetilde{S}$. Lemma 13.5 gives

$$E(Y_S^2) \leqslant E(Y_{\widetilde{S}}^2) \leqslant \sup\big\{E(Y_{S'}^2) \colon S' \in \mathcal{L} \text{ countable}\big\} = E(Y_S^2) \tag{13.25}$$

and, using $E(Y_S(Y_{\widetilde{S}} - Y_S)) = 0$,

$$E\big((Y_S - Y_{\widetilde{S}})^2\big) = 0. \tag{13.26}$$

Hence $Y_{\widetilde{S}} = Y_S$ a.s. and

$$E\big(1_A M\big) = E\big(1_A \mathrm{e}^{Y_S - \frac{1}{2}E(Y_S^2)}\big). \tag{13.27}$$

As this holds for all $A \in \mathcal{F}_T$ and as Y_S and M are $\mathcal{F}_T$-measurable, we get

$$M = E(M|\mathcal{F}_T) = \mathrm{e}^{Y_S - \frac{1}{2}E(Y_S^2)} \text{ a.s.} \tag{13.28}$$

It follows that $T \in \mathcal{L}$ as desired.

To complete the proof, let $\widetilde{P}(A) := E(1_A M)$ and recall that X under $\widetilde{P}$ has the law of $X + f$ under P. Writing $\widetilde{E}$ for the expectation with respect to $\widetilde{P}$,

$$\forall t \in T \colon\; f(t) = \widetilde{E}(X_t) = E(X_t M) = E\Big(X_t\, \mathrm{e}^{Y_S - \frac{1}{2}E(Y_S^2)}\Big) \tag{13.29}$$

For centered Gaussians, the Gaussian integration by parts formula reads $E(Xf(Y)) = \mathrm{Cov}(X,Y)E(f'(Y))$ whenever $f'(Y) \in L^1$. Hence we get $f(t) = \mathrm{Cov}(X_t, Y_S) = \phi_t(Y_S)$. This proves $f \in \mathrm{CMS}(X)$ and $Y_S = \phi^{-1}(f)$ as claimed. □

We leave it to the reader to check that Theorem 13.1 subsumes Theorem 10.4 provided we supplement this with the characterization of the Gaussian Hilbert Space of standard Brownian motion in Lemma 11.5 and the associated Cameron-Martin space in Lemma 11.10. The Cameron-Martin theorem is a spring board to a theory that studies Gaussian measures on Banach spaces using the geometric properties of the Banach space while employing methods of infinite-dimensional vector calculus.