12. Karhunen-Loève expansions

In this chapter we address expansions that are naturally induced by the Hilbert space structure of the Cameron-Martin space. These expansions bear the name of K. Karhunen and M. Loève, who developed these in the context of mathematical statistics.

12.1 Hilbert space expansion.

The Hilbert space structure of the Cameron-Martin space naturally induces a representation of the underlying Gaussian process as a convergent infinite series:

Lemma 12.1 (Karhunen-Loève expansion) *Let $X = \{X_t : t \in T\}$ be a mean-zero Gaussian process and let $\{e_\alpha : \alpha \in \mathcal{I}\}$ be an orthonormal basis in* CMS(X)*. Set $Z_\alpha := \phi^{-1}(e_\alpha)$. Then*

$$\{Z_\alpha : \alpha \in \mathcal{I}\} \text{ are i.i.d. } \mathcal{N}(0,1) \tag{12.1}$$

and

$$\forall t \in T: \quad \sum_{\alpha \in \mathcal{I}} e_\alpha(t)^2 < \infty. \tag{12.2}$$

In particular, the set $\{\alpha \in \mathcal{I} : |e_\alpha(t)| > \epsilon\}$ is finite for each $\epsilon > 0$ and at most countably infinite for $\epsilon = 0$. Moreover,

$$\forall t \in T: \quad X_t = \sum_{\alpha \in \mathcal{I}} e_\alpha(t) Z_\alpha \quad \text{in } L^2 \tag{12.3}$$

and

$$X_t = \lim_{\epsilon \downarrow 0} \sum_{\alpha : |e_\alpha(t)| > \epsilon} e_\alpha(t) Z_\alpha \quad \text{a.s.} \tag{12.4}$$

Proof. The fact that ϕ is a bijective isometry of GHS(X) onto CMS(X) implies that $\{Z_\alpha : \alpha \in \mathcal{I}\}$ is an orthonormal basis in GHS(X). This gives (12.1) and also that

$$\forall Y \in \text{GHS}(X): \quad \sum_{\alpha \in \mathcal{I}} \bigl[E(YZ_\alpha)\bigr]^2 < \infty \ \wedge \ Y = \sum_{\alpha \in \mathcal{I}} E(YZ_\alpha) Z_\alpha \ \text{ in } L^2. \tag{12.5}$$

In particular, the set $\{\alpha \in \mathcal{I} : |E(YZ_\alpha)| > \epsilon\}$ is finite for all $\epsilon > 0$ and, by taking unions, at most countably finite for $\epsilon = 0$. Ordering the countable set $\{\alpha \in \mathcal{I} : |E(YZ_\alpha)| > 0\}$ decreasingly by element size, the resulting infinite series converges also a.s. by Kolmogorov's Three-Series Theorem thanks to $\sum_{\alpha \in \mathcal{I}} [E(YZ_\alpha)]^2 < \infty$.

For the case $Y := X_t$, we get

$$E(X_t Z_\alpha) = \phi_t(Z_\alpha) = e_\alpha(t) \tag{12.6}$$

and so (12.5) implies (12.2–12.3). $\square$

The formulas (12.3) and (12.4) give us a representation of X at a single point but this may not be sufficient for many applications. A natural question arises whether more can be said when X is assumed to be continuous. This is the case provided we also assume that T is compact (if not, then restrict to compact subsets of T):

Lemma 12.2 *Suppose X is a continuous mean-zero Gaussian process on a compact space T. Let $\{e_\alpha\colon \alpha\in\mathcal{I}\}$ be an orthonormal basis in* CMS(X)*. Then each e_α is a bounded function, $\mathcal{I}$ is at most countable and, enumerating $\mathcal{I}=\{\alpha_k\}_{k\geqslant 1}$,*

$$\lim_{n\to\infty}\sup_{t\in T}\sum_{k>n}e_{\alpha_k}(t)^2=0 \tag{12.7}$$

In particular, the infinite series (12.2) *converges uniformly in t.*

Proof. Since X is continuous and T is separable, GHS(X) is separable and so $\mathcal{I}$ is necessarily at most countable. As $t\mapsto E(X_t^2)$ is continuous the compactness of T implies that $\sup_{t\in T}E(X_t^2)<\infty$. The same then applies to $e_\alpha(t)=E(X_tZ_\alpha)$. Parseval's Theorem reads

$$\sum_{k\geqslant 1}e_{\alpha_k}(t)^2=E(X_t^2)<\infty. \tag{12.8}$$

Denoting $\varphi_n(t):=\sum_{k\geqslant n}e_{\alpha_k}(t)^2$, we thus have $\varphi_n(t)\downarrow 0$ for each $t\in T$. Since φ_n is also continuous, the convergence is uniform by Dini's Theorem. □

As a consequence in Lemma 12.2 and the representation (12.3) we get:

Corollary 12.3 *For X a continuous mean-zero Gaussian process on a compact index sets T,*

$$\sup_{t\in T}E\left(\Big|X_t-\sum_{k=0}^n e_{\alpha_k}(t)Z_{\alpha_k}\Big|^2\right)\underset{n\to\infty}{\longrightarrow}0 \tag{12.9}$$

In short, the convergence in (12.3) *is uniform in L^2.*

Proof. The Plancherel formula along with (12.3) gives

$$E\left(\Big|X_t-\sum_{k=0}^n e_{\alpha_k}(t)Z_{\alpha_k}\Big|^2\right)=\sum_{k>n}e_{\alpha_k}(t)^2 \tag{12.10}$$

This tends to zero uniformly in t by above lemma. □

12.2 Convergence in $C(T)$.

While it is tempting to think of (12.3) as an expansion of X into the basis $\{e_\alpha\colon \alpha\in\mathcal{I}\}$, this is valid only when $\mathcal{I}$ (and thus, effectively, T) is finite. Indeed, for $\mathcal{I}$ infinite we get $\sum_{\alpha\in\mathcal{I}}Z_\alpha^2=\infty$ a.s. and so one cannot think of $\{Z_\alpha\colon\alpha\in\mathcal{I}\}$ as (random) coefficients in $\ell^2(\mathcal{I})$. As we show later , this can be mended by noting that $\{e_\alpha\colon\alpha\in\mathcal{I}\}$ are naturally members of a Banach space $\mathcal{B}$ that embeds CMS(X) as a dense subset and for which $\sum_{\alpha\in\mathcal{I}}Z_\alpha e_\alpha$ converges in $\mathcal{B}$, a.s. While this procedure can be applied to any Gaussian Hilbert space, for us the typical choice of $\mathcal{B}$ will be the space $C(T)$.

To demonstrate the above on an example, consider the Lévy construction of Brownian motion. There we took the Haar basis $\{f_k\}_{k\in\mathbb{N}}$ of functions in $L^2([0,t])$ and defined $\{B_u\colon u\leqslant t\}$ as the a.s.-uniform limit

$$B_u:=\lim_{n\to\infty}\sum_{k=1}^n\Big(\int_0^u f_k(s)\mathrm{d}s\Big)Z_k \tag{12.11}$$

for $\{Z_k\colon k \geqslant 1\}$ i.i.d. $\mathcal{N}(0,1)$. The functions $h_n(u) := \int_0^u f_n(s)\mathrm{d}s$ form an orthonormal basis — the so called Schauder functions — in the Cameron-Martin space $\mathrm{CMS}(B)$, which is a dense subset of the Banach space $\mathcal{B} := C([0,t])$ endowed with the supremum norm. The convergence in (12.11) is then uniform a.s. and Z_n thus aquires the meaning of the coefficient associated with the basis element h_n in $C([0,t])$.

A natural question arising in the Lévy construction is whether the uniform convergence is special to the Haar/Schauder basis or whether the same applies more generally. We will settle this with the help of:

Theorem 12.4 *Let $\{X_t\colon t \in T\}$ be a continuous mean-zero Gaussian process with T compact. Then for any orthonormal basis $\{e_n\colon n \geqslant 1\}$ in* $\mathrm{CMS}(X)$ *and $Z_n := \phi^{-1}(e_n)$,*

$$\lim_{n\to\infty} \sup_{t\in T} \Bigl|X_t - \sum_{k=1}^n e_k(t) Z_k\Bigr| = 0 \quad \text{a.s.} \tag{12.12}$$

In short, the convergence in (12.3) *is uniform in $t \in T$ a.s.*

The proof hinges on a rather non-trivial theorem from convergence theory of Banach-space valued random series:

Theorem 12.5 (Itô-Nishio Theorem) *Let $\mathcal{B}$ be a Banach space and let $\{X_n\}_{n\geqslant 1}$ be independent $\mathcal{B}$-valued random variables with $X_n \overset{\text{law}}{=} -X_n$ for all $n \geqslant 1$. Writing $\mathcal{B}^\star$ for the space of continuous linear functionals on $\mathcal{B}$, the following are equivalent:*

(1) *$\{\sum_{k=1}^n X_k\}_{n\geqslant 1}$ converges as $n \to \infty$ in norm of $\mathcal{B}$ a.s.,*
(2) *$\forall \varphi \in \mathcal{B}^\star\colon\ \{\varphi(\sum_{k=1}^n X_k)\}_{n\geqslant 1}$ converges in law.*

While we will not prove this theorem here, let us recall that it generalizes the following result due to P. Lévy:

Lemma 12.6 *Let $\{X_n\}_{n\geqslant 1}$ be independent and let $S_n := X_1 + \cdots + X_n$. Then*

(1) *S_n converges a.s.,*
(2) *S_n converges in probability,*
(3) *S_n converges in law.*

are all equivalent.

We also note that this lemma immediately generalizes to $\mathbb{R}^d$-valued random variables. Indeed, thanks to the Cramér-Wold device, S_n converging in law is equivalent to $v \cdot S_n$ converging in law for each $v \in \mathbb{R}^d$. The lemma then makes it equivalent to $v \cdot S_n$ converging a.s. for each $v \in \mathbb{R}^d$. Specializing to v in coordinate directions, each coordinate of S_n then converges a.s., proving a.s. convergence of S_n. Since $\varphi(x) := v \cdot x$ is a general form of a linear functional on $\mathbb{R}^d$, this gives Theorem 12.5 for any $\mathcal{B}$ with $\dim(\mathcal{B}) < \infty$. The case of infinite dimension is harder to prove and, moreover, requires the additional assumption of symmetry.

We will now move to the proof of uniform convergence of the Karhunen-Loève expansion to the underlying Gaussian process:

Proof of Theorem 12.4 assuming the Itô-Nishio Theorem. By Theorem 12.5 it suffices to show

$$\forall \varphi \in C(T)^\star\colon\ \varphi\Bigl(\sum_{k>n} e_k Z_k\Bigr) \xrightarrow[n\to\infty]{L^1} 0. \tag{12.13}$$

Indeed, the continuity and additivity of φ then implies that $\varphi(X - \sum_{k=1}^n e_k Z_k)$ converges in law and so $X - \sum_{k=1}^n e_k Z_k$ converges in the supremum norm a.s.

To get (12.13), we note that the Riesz representation theorem shows that each $\varphi \in C(T)^\star$ takes the form $\varphi(f) = \int f \mathrm{d}\mu$ for a finite signed (Borel) measure μ on T. The Cauchy-Schwarz inequality then gives

$$E\biggl|\varphi\Bigl(\sum_{k>n} e_k Z_k\Bigr)\biggr| \leqslant \int_T E\Bigl|\sum_{k>n} e_k(t) Z_k\Bigr|\, |\mu|(\mathrm{d}t) \leqslant \int_T \sqrt{\sum_{k>n} e_k(t)^2}\, |\mu|(\mathrm{d}t). \tag{12.14}$$

By (12.7), the square root tends to zero as $n \to \infty$ uniformly in t and, since $|\mu|$ is a finite measure, we get (12.13) by the Bounded Convergence Theorem. (We actually only used that $\sup_{k\geqslant 1} \sum_{k\geqslant 1} e_k(t)^2 < \infty$ and that $\sum_{k>n} e_k(t)^2 \to 0$ pointwise. But compactness, which implies uniform convergence, is used in other places in this proof as well.) □

As a corollary, we answer the question posed above:

Corollary 12.7 *Given any $t > 0$, let $\{f_k\}_{k\in\mathbb{N}}$ be any orthonormal basis in $L^2([0,t])$ and let $\{Z_n\}_{n\in\mathbb{N}}$ be i.i.d. $\mathcal{N}(0,1)$. Then*

$$\sum_{k=0}^{\infty} \Bigl(\int_0^u f_k(s)\mathrm{d}s\Bigr) Z_k \tag{12.15}$$

converges uniformly in $u \in [0,t]$, a.s. The limit function has the law of standard Brownian motion on $[0,t]$, modulo a modification on a null set.

Proof. Consider a standard Brownian motion $\{B_u\colon u \leqslant t\}$ and realize the sequence $\{Z_k\}_{k\in\mathbb{N}}$ by $Z_k := \int_0^t f_k(s)\mathrm{d}B_s$. Indeed, Itô isometry shows that $\{Z_k\colon k \in \mathbb{N}\}$ are i.i.d. $\mathcal{N}(0,1)$ and, as shown earlier, $e_k(u) := \int_0^u f_k(s)\mathrm{d}s$ defines an ONB in $\mathrm{CMS}(B)$. Theorem 12.4 then gives $\sum_{k=0}^n e_k(u) Z_k \to B_u$ uniformly in $u \in [0,t]$ a.s., as desired. □

We note that the corollary seems hard to prove "by hand" given how the specific properties of the Haar functions enter the original Lévy construction. On the other hand, the proof does use that the limit has a continuous version in at least one basis. (We need this because Theorem 12.4 is formulated for continuous processes.)