11. Gaussian Hilbert space and the Cameron-Martin space

We will now move to a more abstract level and present parts of a rich theory of Gaussian processes over arbitrary index sets. Much of what we do actually applies to general square integrable processes but, as we will see in the forthcoming lectures, the Gaussian case adds a structure that will ultimately make these particularly elegant.

11.1 Gaussian Hilbert space.

We recall the following definition:

Definition 11.1 *Given a set T, a* Gaussian process on T *is a family* $\{X_t : t \in T\}$ *of* $\mathbb{R}$-*valued random variables on a probability space such that, for any finite* $S \subseteq T$ *and any* $\lambda : S \to \mathbb{R}$*, the random variable* $\sum_{t \in S} \lambda(t) X_t$ *has normal law.*

Note that the above means that, in particular, every X_t is normal. We will focus on mean zero processes which are those for which $\forall t \in T\colon\ EX_t = 0$. The standard Brownian motion and white noise are examples of such a family. Next we put forward:

Definition 11.2 *Given a mean-zero Gaussian process* $\{X_t : t \in T\}$ *on* $(\Omega, \mathcal{F}, P)$*, the associated* Gaussian Hilbert space $\mathrm{GHS}(X)$ *is defined by*

$$\mathrm{GHS}(X) := \overline{\mathrm{span}\{X_t : t \in T\}}^{L^2(\Omega,\mathcal{F},P)} \tag{11.1}$$

Here we noted that any linear combination of random variables from $\{X_t : t \in T\}$ lies in L^2 and so we can close the set in L^2-topology. We now observe:

Lemma 11.3 *Under the topology of L^2-convergence,* $\mathrm{GHS}(X)$ *is a closed linear vector space of random variables with mean-zero (multivariate) Gaussian law.*

Proof. Linearity and closedness in L^2 are consequence of the definition. The fact that the joint law is multivariate Gaussian follows from this property being preserved under linear combinations and L^2-limits. □

We note that the properties listed in Lemma 11.3 define the notion of a Gaussian Hilbert space $\mathcal{H}$ regardless of an underlying Gaussian process. If we index such an $\mathcal{H}$ by itself, it becomes a Gaussian process with an additional additivity/continuity structure. On the other hand, the linear span of any square-integrable random variables lies in L^2 and so the above can be considered whenever $\{X_t : t \in T\} \subseteq L^2(\Omega, \mathcal{F}, P)$.

Let us check some elementary examples:

Lemma 11.4 *Let $T = \{1, \dots, n\}$ and assume a mean-zero Gaussian vector $\{X_k : k \in T\}$ is given with covariance $C = \{\mathrm{Cov}(X_i, X_j)\}_{i,j=1}^n$. Then* $\mathrm{GHS}(X)$ *is the set of* $\mathbb{R}^n$*-valued mean-zero Gaussian random vectors taking values in* $\mathrm{Ran}(C) = \mathrm{Ker}(C)^\perp$.

Leaving a proof of this to homework exercise, we move on to:

Lemma 11.5 *Let $B = \{B_s \colon s \leqslant t\}$ be a standard Brownian motion on $[0,t]$. Then*

$$\mathrm{GHS}(B) = \Bigl\{ \int_0^t f(s)\mathrm{d}B_s \colon f \in L^2([0,t]) \Bigr\} \tag{11.2}$$

where $L^2([0,t])$ is the space of (equivalence classes of) Borel functions $f \colon [0,t] \to \mathbb{R}$ that are square integrable w.r.t. the Lebesgue measure on $[0,t]$.

Proof. Let $\mathcal{H}$ denote the set on the right of (11.2). Linearity of the integral along with Itô isometry ensure that $\mathcal{H}$ is a closed linear subspace of L^2. Since $B_u = \int_0^t 1_{[0,u]}(s)\mathrm{d}B_s$, we have $\{B_s \colon s \leqslant t\} \subseteq \mathcal{H}$ and so $\mathrm{GHS}(B) \subseteq \mathcal{H}$. On the other hand, write $\mathcal{H}_0$ for the subset of $\mathcal{H}$ associated with piecewise constant f — i.e., those of the form $f(s) = \sum_{i=1}^n a_i 1_{[0,t_i)}(s)$ for some $a_1, \dots, a_n$ and $t_1, \dots, t_n \geqslant 0$. Such functions are dense $L^2([0,t])$ and so, using Itô isometry again, $\mathcal{H}_0$ is dense $\mathcal{H}$. The observation

$$\int_0^t \Bigl(\sum_{i=1}^n a_i 1_{[0,t_i)}(s) \Bigr) \mathrm{d}B_s = \sum_{i=1}^n a_i B_{t_i \wedge t} \tag{11.3}$$

shows $\mathcal{H}_0 \subseteq \mathrm{GHS}(B)$ and, since $\mathrm{GHS}(B)$ is closed, we get $\mathcal{H} \subseteq \mathrm{GHS}(B)$ as desired. □

The previous lemma is actually a special instance of the next lemma whose proof is executed very much the same way and is thus left to an exercise:

Lemma 11.6 *Given a measure space $(\mathscr{X}, \mathcal{G}, \mu)$ with μ finite, let $W = \{W(A) \colon A \in \mathcal{G}\}$ be a Gaussian white noise. Then*

$$\mathrm{GHS}(W) = \Bigl\{ \int f \mathrm{d}W \colon f \in L^2(\mathscr{X}, \mathcal{G}, \mu) \Bigr\} \tag{11.4}$$

where $\int f \mathrm{d}W$ is the Paley-Wiener integral.

11.2 Cameron-Martin space.

Moving on with general theory, we now observe that a Gaussian process is naturally associated with a function space over the underlying index set by way of the autocorrelation map $t \mapsto \phi_t(Y)$ where

$$\phi_t(Y) := E(X_t Y) \tag{11.5}$$

A key starting observation in this is that this map is injective:

Lemma 11.7 *We have*

$$\forall Y, \widetilde{Y} \in \mathrm{GHS}(X) \colon \;\; \phi(Y) = \phi(\widetilde{Y}) \;\Rightarrow\; Y = \widetilde{Y} \tag{11.6}$$

Proof. Suppose that $\phi(Y) = \phi(\widetilde{Y})$, which means that $\forall t \in T \colon\; \phi_t(Y) = \phi_t(\widetilde{Y})$. This implies that $E(X_t(Y - \widetilde{Y})) = 0$ for all $t \in T$ and, by linearity and continuity in L^2,

$$\forall Z \in \mathrm{GHS}(X) \colon \;\; E\bigl(Z(Y - \widetilde{Y})\bigr) = 0 \tag{11.7}$$

Taking $Z := Y - \widetilde{Y}$ shows $E((Y - \widetilde{Y})^2) = 0$ and thus $Y = \widetilde{Y}$. □

We collect the set of functions obtained via (11.5) into one object:

Definition 11.8 *Given a zero-mean Gaussian process* $X = \{X_t : t \in T\}$,

$$\mathrm{CMS}(X) := \{\phi(Y) : Y \in \mathrm{GHS}(X)\} \tag{11.8}$$

is the Cameron-Martin space *associated with* X.

Another key observation is then:

Lemma 11.9 *Define* $\langle \cdot, \cdot \rangle : \mathrm{CMS}(X) \times \mathrm{CMS}(X) \to \mathbb{R}$ *by*

$$\forall f, g \in \mathrm{CMS}(X) : \ \langle f, g \rangle := E(\phi^{-1}(f), \phi^{-1} g) \tag{11.9}$$

Then $f, g \mapsto \langle f, g \rangle$ *is an inner product on* $\mathrm{CMS}(X)$ *and, using the norm topology associated therewith, the map* ϕ *is an isometric bijection of* $\mathrm{GHS}(X)$ *onto* $\mathrm{CMS}(X)$.

Proof. That ϕ is injective was checked in Lemma 11.7 and that it is onto is a consequence of the definition (11.8). The space $\mathrm{GHS}(X)$ inherits the natural inner product $Y, Z \mapsto E(YZ)$ from L^2 and ϕ preserves it by definition. It follows that ϕ is an isometry. □

We again remark that the above works equally well when $\{X_t : t \in T\}$ is just a subset of $L^2(\Omega, \mathcal{F}, P)$. In this case we would not attach the names of Cameron and Martin to the resulting function space.

Moving to examples, we note:

Lemma 11.10 *Let* $B = \{B_s : s \leqslant t\}$ *be a standard Brownian motion on* $[0, t]$. *Then*

$$\mathrm{CMS}(B) = \Big\{ F \in \mathrm{AC}([0,t]) : F(0) = 0 \ \wedge \ F' \in L^2([0,t]) \Big\} \tag{11.10}$$

where $\mathrm{AC}([0,t])$ *is the space of absolutely-continuous functions* $[0,t] \to \mathbb{R}$. *The canonical inner product then takes the form*

$$\forall F, G \in \mathrm{CMS}(B) : \ \langle F, G \rangle = \int_0^t F'(s) G'(s) \mathrm{d}s \tag{11.11}$$

Proof. From Lemma 11.5 we know that elements of $\mathrm{GHS}(B)$ are Paley-Wiener integrals of the form $\int_0^t f(s) \mathrm{d}B_s$. The Itô isometry and polarization identity show

$$\begin{aligned}
\phi_u\Big(\int_0^t f(s) \mathrm{d}B_s\Big) &= E\Big(B_u \int_0^t f(s) \mathrm{d}B_s\Big) \\
&= E\Big(\Big(\int_0^t 1_{[0,u]}(s) \mathrm{d}s\Big)\Big(\int_0^t f(s) \mathrm{d}B_s\Big)\Big) \\
&= \int_0^t 1_{[0,u]}(s) f(s) \mathrm{d}s = \int_0^u f(s) \mathrm{d}s
\end{aligned} \tag{11.12}$$

This shows that

$$\mathrm{CMS}(B) = \Big\{ u \mapsto \int_0^u f(s) \mathrm{d}s : f \in L^2([0,t]) \Big\} \tag{11.13}$$

As, by the Vitali-Lebesgue theorem, $F \in \mathrm{AC}([0,t])$ is equivalent to F' being defined a.e. and $F(u) - F(0) = \int_0^u F'(s) \mathrm{d}s$, this gives (11.10).

For the second part, notice that (11.12) applied inside (11.9) gives

$$\begin{aligned}\langle F,G\rangle &= \left\langle \int_0^{\cdot} F'(s)\mathrm{d}s,\ \int_0^{\cdot} G'(s)\mathrm{d}s \right\rangle \\ &= E\left(\left(\int_0^t F'(s)\mathrm{d}B_s\right)\left(\int_0^t G'(s)\mathrm{d}B_s\right)\right) = \int_0^t F'(s)G'(s)\mathrm{d}s,\end{aligned} \tag{11.14}$$

where the Itô isometry along with polarization identity was used in the last step. □

Again, a very similar argument applies to the white noise process as well:

Lemma 11.11 *Denote by $\mathcal{M}(\mathscr{X},\mathcal{G})$ the set of finite signed measures on a measurable space $(\mathscr{X},\mathcal{G})$ with μ finite. Then for a white noise $W = \{W(A)\colon A \in \mathcal{G}\}$ over $(\mathscr{X},\mathcal{G},\mu)$, we have*

$$\mathrm{CMS}(W) = \left\{ \nu \in \mathcal{M}(\mathscr{X},\mathcal{G})\colon \nu \ll \mu \wedge \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^2(\mathscr{X},\mathcal{G},\mu) \right\} \tag{11.15}$$

with

$$\forall \nu,\tilde{\nu} \in \mathrm{CMS}(W)\colon\ \langle \nu,\tilde{\nu}\rangle = \int \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\frac{\mathrm{d}\tilde{\nu}}{\mathrm{d}\mu}\mathrm{d}\mu. \tag{11.16}$$

Proof. Lemma 11.6 gives that elements of $\mathrm{GHS}(W)$ are Paley-Wiener integrals of the form $\int f\mathrm{d}W$ where $f \in L^2(\mu)$. Hence

$$\phi_A\left(\int f\mathrm{d}W\right) := E\left(W(A)\int f\mathrm{d}W\right) = \int_A f\mathrm{d}\mu \tag{11.17}$$

by the isometry defining the Paley-Wiener integral. As $L^2(\mu) \subseteq L^1(\mu)$ due to μ being finite, it follows that $A \mapsto \phi_A$ is a finite signed measure on $(\mathscr{X},\mathcal{G})$.

Writing $\nu(A) := \int_A f\mathrm{d}\mu$ shows "$\supseteq$" in (11.15). For the opposite inclusion we pick a measure ν with $\nu \ll \mu$, denote the Radon-Nikodym derivative by f and use $f \in L^2$ to infer $\int f\mathrm{d}W \in \mathrm{GHS}(X)$. Now apply (11.17) one more time. The identity (11.16) is proved exactly the same way as (11.11); we leave the details to the reader. □

In both of the examples above, the Cameron-Martin space consisted of continuous functions. (For measures, the continuity is w.r.t. pseudometric $A,B \mapsto \mu(A\triangle B)$.) This is not a coincidence, as our next lemma shows:

Lemma 11.12 *Suppose T is a metric space and the sample paths of a mean-zero Gaussian process $\{X_t\colon t \in T\}$ are continuous a.s. or at least continuous in L^2. Then*

$$\mathrm{CMS}(X) \subseteq C(T) \tag{11.18}$$

Proof. Let $Y \in \mathrm{GHS}(X)$. We need to show that if $t_n \to$ in T then $\phi_{t_n}(Y) \to \phi_t(Y)$. That will follow if $X_{t_n} \to X_t$ in L^2 which is either assumed directly in the statement or is derived from the assumption of a.s.-continuity X by noting that, by the Gaussian nature of the random variables, $X_{t_n} - X_t \to 0$ in distribution implies $\mathrm{Var}(X_{t_n} - X_t) \to 0$ which is equivalent to $X_{t_n} \to X_t$ in L^2. □

We remark that, as noted earlier, even here the assumption that X is Gaussian is not strictly used. All that we need is that $\{X_t\colon t \in T\} \subseteq L^2(\Omega,\mathcal{F},P)$.