10. Shifted Brownian paths

In this section we change the point of view somewhat and discuss the celebrated Cameron-Martin theorem in the context of the Wiener measure. The main result will be arrived at organically and is stated at the very end of this chapter.

10.1 The Cameron-Martin problem and sufficient conditions.

While we introduced the Wiener measure as the canonical law of the standard Brownian motion, from the perspective of integration theory it is just a probability measure on the space $C[0,\infty)$. A natural notion there is that of a *shift*, which is a map $\omega \mapsto \omega_f$ that takes $\omega \in C[0,\infty)$ and assigns it $\omega_f(t) := \omega(t) + f(t)$, for some $f\colon [0,\infty) \to \mathbb{R}$.

In 1944, R.H. Cameron and W.T. Martin asked the following question: For what functions f is the push forward of the Wiener measure by $\omega \mapsto \omega_f$ absolutely continuous with respect to the Wiener measure? They characterized the class of functions for which this is the case and even computed the Radon-Nikodym derivative.

It is clear that we need to make f continuous so that the image of $\omega \mapsto \omega_f$ even lies in $C[0,\infty)$. Since the Wiener measure concentrates on functions that vanish at zero, we also need $f(0) = 0$. But what other kinds of regularity may be needed? The following lemma, drawing heavily on Girsanov's theorem (which did not exist at the time Cameron and Martin did their work), gives a sufficient condition:

Lemma 10.1 (Sufficient condition for absolute continuity) *Let B be the standard Brownian motion with $B_0 = 0$ realized by canonical coordinate projections on $C[0,\infty)$. Fix $t > 0$ and set $\mathcal{F}_t^B := \sigma(B_s\colon s \leqslant t)$. Then for all functions $f \in C[0,\infty)$ with $f(0) = 0$,*

$$f \in \mathrm{AC}[0,t] \;\wedge\; f' \in L^2([0,t]) \tag{10.1}$$

implies

$$P(B + f \in \cdot)|_{\mathcal{F}_t^B} \ll P(B \in \cdot)|_{\mathcal{F}_t^B} \tag{10.2}$$

and

$$\frac{\mathrm{d}P(B+f\in\cdot)|_{\mathcal{F}_t^B}}{\mathrm{d}P(B\in\cdot)|_{\mathcal{F}_t^B}} = \exp\left\{\int_0^t f'(s)\mathrm{d}B_s - \frac{1}{2}\int_0^t f'(s)^2\mathrm{d}s\right\} \quad \text{a.s.} \tag{10.3}$$

The measures in (10.2) *are actually equivalent.*

Proof. The assumption $f \in \mathrm{AC}[0,t]$ along with $f(0) = 0$ implies that $f(u) = \int_0^u h(s)\mathrm{d}s$ for some $h \in L^1[0,t]$. The Lebesgue differentiation theorem then shows that f is actually a.e. differentiable with $f' = h$ a.e. Thanks to Novikov's condition (in fact, even just Lemma 9.2 suffices), the condition $f' \in L^2[0,t]$ ensures that

$$M_u := \exp\left\{\int_0^{u\wedge t} f'(s)\mathrm{d}B_s - \frac{1}{2}\int_0^{u\wedge t} f'(s)^2\mathrm{d}s\right\} \tag{10.4}$$

is a martingale and so, in particular, $EM_t = 1$. Setting $\widetilde{P}(A) := E(1_A M_t)$ for $A \in \mathcal{F}_t^B$, we get a probability measure $\widetilde{P}$ for which Girsanov's theorem gives

$$\left\{B_u - \int_0^u f'(s)\mathrm{d}s\colon u \in [0,t]\right\} \text{ under } \widetilde{P} \overset{\text{law}}{=} \{B_u\colon u \in [0,t]\} \text{ under } P. \tag{10.5}$$

Since $\int_0^u f'(s)\mathrm{d}s = f(u)$, we can write this shortly as

$$\forall A \in \mathcal{F}_t^B\colon\ \widetilde{P}(B - f \in A) = P(B \in A). \tag{10.6}$$

Replacing A by $\{\omega - f\colon \omega \in A\}$ converts this into

$$\forall A \in \mathcal{F}_t^B\colon\ \widetilde{P}(B \in A) = P(B + f \in A). \tag{10.7}$$

Using the explicit form of $\widetilde{P}$, we thus get

$$\forall A \in \mathcal{F}_t^B\colon\ P(B + f \in A) = E\big(1_{\{B\in A\}} M_t\big). \tag{10.8}$$

But this gives $P(B + f \in A) = 0$ whenever $P(B \in A) = 0$, proving (10.2), with the Radon-Nikodym derivative of the two measures equal to M_t a.s, proving also (10.38). Since the Radon-Nikodym derivative is non-vanishing a.s., the measures are equivalent. $\square$

A subtle step of the proof is the conversion from (10.6) to (10.7). This is exactly what precludes direct extension of the statement beyond shifts by deterministic functions. To demonstrate what changes when we attempt to proceed along that argument anyway, we notice the following corollary of Girsanov's theorem:

Lemma 10.2 *Consider the Wiener space with the standard Brownian motion B obtained by the coordinate projection. Let $a\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be Borel measurable such that the SDE*

$$\mathrm{d}X_t = -a(t, X_t)\mathrm{d}t + \mathrm{d}B_t \tag{10.9}$$

admits a unique strong solution on the Wiener space. Let $t > 0$ and suppose the solution obeys the Novikov condition

$$E\bigg(\exp\Big\{\frac{1}{2}\int_0^t a(s, X_s)^2 \mathrm{d}s\Big\}\bigg) < \infty \tag{10.10}$$

Then $\{a(s \wedge t, B_{s\wedge t})\colon s \geqslant 0\} \in \mathcal{V}_B^{\mathrm{loc}}$ and $W_u := B_u + \int_0^{u\wedge t} a(s, B_s)\mathrm{d}s$ obeys

$$P(W \in \cdot)|_{\mathcal{F}_t^B} \ll P(B \in \cdot)|_{\mathcal{F}_t^B} \tag{10.11}$$

and

$$\frac{\mathrm{d}P(W \in \cdot)|_{\mathcal{F}_t^B}}{\mathrm{d}P(B \in \cdot)|_{\mathcal{F}_t^B}} = \exp\bigg\{\int_0^t a(s, X_s)\mathrm{d}B_s - \frac{1}{2}\int_0^t a(s, X_s)^2 \mathrm{d}s\bigg\} \quad \text{a.s.} \tag{10.12}$$

The right-hand side is meaningful as a function on Wiener space because, by strong uniqueness, the solution X is a measurable function of the Brownian path via the solution map.

Proof. Given the strong solution as above, let

$$M_u := \exp\bigg\{\int_0^{u\wedge t} a(s, X_s)\mathrm{d}B_s - \frac{1}{2}\int_0^{u\wedge t} a(s, X_s)^2 \mathrm{d}s\bigg\} \tag{10.13}$$

The condition (10.10) with the help of Theorem 9.3 ensures that M is a martingale and $\widetilde{P}$ defined for $A \in \mathcal{F}_t^B$ by $\widetilde{P}(A) := E(1_A M_t)$ is a probability measure. Moreover, since $X_u = B_u - \int_0^u a(s, X_s)\mathrm{d}s$ for $u \in [0, t]$, Girsanov's theorem tells us that $\{X_u\colon u \leqslant t\}$ is a standard

Brownian motion under $\widetilde{P}$. Since $\int_0^t a(s,X_s)^2\mathrm{d}s < \infty$ P-a.s., the absolute continuity $\widetilde{P} \ll P$ on $\mathcal{F}_t^B$ implies $\int_0^t a(s,B_s)^2\mathrm{d}s < \infty$ P-a.s. and so W is well defined. Moreover, we get

$$\forall A \in \mathcal{F}_t^B\colon\ P(W \in A) = P\left(B + \int_0^\cdot a(s,B_s)\mathrm{d}s \in A\right) \\ = \widetilde{P}\left(X + \int_0^\cdot a(s,X_s)\mathrm{d}s \in A\right) = \widetilde{P}(B \in A) = E\big(1_{\{B\in A\}}M_t\big). \tag{10.14}$$

This now readily implies (10.11–10.12). □

The above naturally subsumes the claim in Lemma 10.1 by putting $a(s,x) := f'(s)$ and noting that then $W = B + f$. The statement also hightlights what is different once the integrand depends on the process itself: The change of measure replaces the Brownian motion inside the argument by the solution to an SDE. This seems to suggest that, if the integrand depends on more than just the instantaneous value of the process, e.g., if takes the form of a function of the form $s \mapsto a(s,\{B_u\colon u \leqslant s\})$, then we need to solve

$$\mathrm{d}X_t = a\big(t,\{X_u\colon u \leqslant t\}\big)\mathrm{d}t + \mathrm{d}B_t. \tag{10.15}$$

This is hard to do directly due to non-locality of the dependence but we may later show a way to do this using the Itô chaos decomposition.

10.2 Necessary condition and the Cameron-Martin theorem.

As it turns out, the sufficient condition expressed in Lemma 10.1 is actually necessary. This is easy to state but the proof is more involved:

Lemma 10.3 *Let $f\colon [0,\infty) \to \mathbb{R}$ be such that $f(0) = 0$ and let $t > 0$. Then*

$$P(B + f \in \cdot)|_{\mathcal{F}_t^B} \ll P(B \in \cdot)|_{\mathcal{F}_t^B} \tag{10.16}$$

implies

$$f \in \mathrm{AC}[0,t] \ \wedge\ f' \in L^2([0,t]). \tag{10.17}$$

Proof. Suppose that (10.16) holds for some $t > 0$ and let F denote the Radon-Nikodym derivative of the two measures. Given any partition $\Pi := \{0 = t_0 < \cdots < t_n = t\}$, observe that then

$$E\big(F \,|\, \sigma(B_{t_1},\dots,B_{t_n})\big) = \frac{\mathrm{d}P((B_{t_1}+f(t_1),\dots,B_{t_n}+f(t_n)) \in \cdot)}{\mathrm{d}P((B_{t_1},\dots,B_{t_n}) \in \cdot)} \tag{10.18}$$

Since $P((B_{t_1},\dots,B_{t_n}) \in \cdot)$ has density

$$f(x_1,\dots,x_n) = \left(\prod_{i=1}^n \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}}\right)\exp\left\{-\frac{1}{2}\sum_{i=1}^n \frac{(x_{t_i} - x_{t_{i-1}})^2}{t_i - t_{i-1}}\right\} \tag{10.19}$$

with respect to the Lebesgue measure, a direct calculation shows that

$$\begin{aligned} &E\big(F \,|\, \sigma(B_{t_1},\dots,B_{t_n})\big) \\ &\qquad = \exp\biggl\{\sum_{i=1}^n \frac{f(t_i)-f(t_{i-1})}{t_i-t_{i-1}}(B_{t_i}-B_{t_{i-1}}) - \frac12\sum_{i=1}^n \frac{[f(t_i)-f(t_{i-1})]^2}{t_i-t_{i-1}}\biggr\} \end{aligned} \tag{10.20}$$

Denoting, for the given partition Π,

$$g_\Pi(s) := \sum_{i=1}^n \frac{f(t_i)-f(t_{i-1})}{t_i-t_{i-1}} 1_{(t_{i-1},t_i]}(s) \tag{10.21}$$

we then have

$$E\big(F \,|\, \sigma(B_{t_1},\dots,B_{t_n})\big) = \exp\biggl\{\int_0^t g_\Pi(s)\mathrm{d}B_s - \frac12\int_0^t g_\Pi(s)^2\mathrm{d}s\biggr\} \tag{10.22}$$

i.e., the Radon-Nikodym derivative has the form of the exponential martingale albeit only with respect to a piece-wise constant function.

Next, consider the dyadic partitions $\Pi_n := \{tk2^{-n} \colon k = 0,\dots,2^n\}$ of $[0,t]$ and observe that these are nested. The corresponding σ-algebras $\{\sigma(B_u \colon u \in \Pi_n)\}$ are thus increasing and, since $\sigma(\bigcup_{n\geqslant1}\sigma(B_u \colon u \in \Pi_n)) = \sigma(B_u \colon u \leqslant t)$ by sample path continuity, the Lévy Forward theorem shows

$$E\big(F \,|\, \sigma(B_{t_1},\dots,B_{t_n})\big) \underset{n\to\infty}{\longrightarrow} F \quad \text{a.s.} \tag{10.23}$$

This means that

$$\sup_{n\geqslant1}\ \exp\biggl\{\int_0^t g_{\Pi_n}(s)\mathrm{d}B_s - \frac12\int_0^t g_{\Pi_n}(s)^2\mathrm{d}s\biggr\} < \infty \quad \text{a.s.} \tag{10.24}$$

and, in light of absolute continuity of the law of $B+f$ with respect to that of B, also

$$\sup_{n\geqslant1}\ \exp\biggl\{\int_0^t g_{\Pi_n}(s)\mathrm{d}(B+f)_s - \frac12\int_0^t g_{\Pi_n}(s)^2\mathrm{d}s\biggr\} < \infty \quad \text{a.s.} \tag{10.25}$$

Noting that the explicit form of g_{Π_n} gives

$$\int_0^t g_{\Pi_n}(s)\mathrm{d}f(s) = \int_0^t g_{\Pi_n}(s)^2\mathrm{d}s, \tag{10.26}$$

this becomes

$$\sup_{n\geqslant1}\ \exp\biggl\{\int_0^t g_{\Pi_n}(s)\mathrm{d}B_s + \frac12\int_0^t g_{\Pi_n}(s)^2\mathrm{d}s\biggr\} < \infty \quad \text{a.s.} \tag{10.27}$$

Since F integrates to one, it has to be non-zero on a set of non-zero measure. On this set we then also have

$$\inf_{n\geqslant1}\ \exp\biggl\{\int_0^t g_{\Pi_n}(s)\mathrm{d}B - \frac12\int_0^t g_{\Pi_n}(s)^2\mathrm{d}s\biggr\} > 0 \quad \text{a.s.} \tag{10.28}$$

As this implies that, for at least one path of B, the exponential in (10.27) remains uniformly bounded while that in (10.28) remains uniformly positive, dividing one by the other cancels the stochastic integral and gives

$$A := \sup_{n\geqslant 1}\int_0^t g_{\Pi_n}(s)^2\mathrm{d}s < \infty \tag{10.29}$$

This is now a statement about a sequence of deterministic functions of one real variable, from which we will derive the claim using analysis.

We start with the proof of absolute continuity of f. Let $[a_1,b_1],\dots,[a_k,b_k]$ be disjoint intervals with endpoint in $\bigcup_{n\geqslant 1}\Pi_n$. Define

$$h(s) := \begin{cases}\operatorname{sgn}(f(b_i)-f(a_i)), & \text{if } s\in[a_i,b_i] \text{ for some } i=\dots,k,\\ 0, & \text{else.}\end{cases} \tag{10.30}$$

Once n is so large that all $a_1,\dots,a_k,b_1,\dots,b_k\in\Pi_n$, we have

$$\sum_{i=1}^k\bigl|f(b_i)-f(a_i)\bigr| = \int h(s)g_{\Pi_n}(s)\mathrm{d}s \tag{10.31}$$

The right-hand side is bounded using the Cauchy-Schwarz inequality with the result

$$\int h(s)g_{\Pi_n}(s) \leqslant \biggl(\int_0^t g_{\Pi_n}(s)^2\mathrm{d}s\biggr)^{1/2}\biggl(\int h(s)^2\mathrm{d}s\biggr)^{1/2} \tag{10.32}$$

Since the last integral equals $\sum_{i=1}^k|b_i-a_i|$, we thus get

$$\sum_{i=1}^k\bigl|f(b_i)-f(a_i)\bigr| \leqslant \sqrt{A}\,\biggl(\sum_{i=1}^k|b_i-a_i|\biggr)^{1/2} \tag{10.33}$$

Given $\epsilon>0$ we now set $\delta := \epsilon^2/A$ and observe that then $\sum_{i=1}^k|b_i-a_i|<\delta$ implies $\sum_{i=1}^k|f(b_i)-f(a_i)|<\epsilon$. As this works for all collections of intervals with endpoints in a dense subset of $[0,t]$, we have shown $f\in \mathrm{AC}[0,t]$.

The absolute continuity ensures that f is Lebesgue differentiable with $f'\in L^1([0,t])$ and $f(u)=\int_0^u f'(s)\mathrm{d}s$ valid for all $u\in[0,t]$ due to $f(0)=0$. One might be tempted to invoke the Lebesgue differentiation theorem to claim $g_{\Pi_n}\to f'$ a.e., but the problem here is that the intervals come from a sequence of partitions which is not compatible with the standard statement of the Lebesgue differentiation theorem. We proceed by a probabilistic argument.

Let U be uniform on $[0,t]$ and let $\mathcal{G}_n := \sigma(\{u\leqslant U\leqslant s\}\colon u,s\in\Pi_n,\ u<s\}$. Then, as shown by a direct calculation,

$$g_{\Pi_n}(U) = E\bigl(f'(U)\,\big|\,\mathcal{G}_n\bigr) \tag{10.34}$$

Since $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ is increasing, the Lévy Forward Theorem along with $\sigma(\bigcup_{n\geqslant 1}\mathcal{G}_n)=\sigma(U)$ gives $g_{\Pi_n}(U)\to f'(U)$ a.s. Fatou's lemma turns this into

$$E\bigl(f'(U)^2\bigr) \leqslant \liminf_{n\to\infty}E\bigl(g_{\Pi_n}(U)^2\bigr) \leqslant \sup_{n\geqslant 1}E\bigl(g_{\Pi_n}(U)^2\bigr) \tag{10.35}$$

Since the expectation with respect to U is $1/t$-multiple of the integral of the function on $[0,t]$, the claim now follows from (10.29). $\square$

To summarize the above developments, we state:

Theorem 10.4 (Cameron and Martin, 1944) *Consider the standard Brownian motion B on the Wiener space such that $B_0 = 0$. Then for all $f\colon [0,\infty) \to \mathbb{R}$ and all $t > 0$,*

$$f \in \mathrm{AC}[0,t] \;\wedge\; f(0) = 0 \;\wedge\; f' \in L^2([0,t]) \tag{10.36}$$

is equivalent to

$$P(B + f \in \cdot)|_{\mathcal{F}_t^B} \ll P(B \in \cdot)|_{\mathcal{F}_t^B} \tag{10.37}$$

and, in fact, also to these measures being mutually equivalent. Under these circumstances,

$$\frac{\mathrm{d}P(B + f \in \cdot)|_{\mathcal{F}_t^B}}{\mathrm{d}P(B \in \cdot)|_{\mathcal{F}_t^B}} = \exp\biggl\{\int_0^t f'(s)\mathrm{d}B_s - \frac{1}{2}\int_0^t f'(s)^2\mathrm{d}s\biggr\} \tag{10.38}$$

almost surely with respect to the Wiener measure.

This result by R.H. Cameron and W.T. Martin, which appeared under the title “Transformations of Weiner Integrals Under Translations” in Annals of Mathematics, Vol. 45, No. 2 (1944), pp. 386-396, started a slew of developments in the theory of Gaussian processes, and later also martingales, that are either referred to by the names of these authors or by Girsanov’s. We will discuss these in the forthcoming lectures.