Existence of continuous version & conditions for integrability

Previously: $\forall Y \in \mathcal{D}_0^{[\cdot]}$ $\forall t \geq 0$ $\exists \int_0^t Y_s \, dB_s$ ... $E\left[\left(\int_0^t Y_s \, dB_s\right)^2\right] = E\left(\int_0^t Y_s^2 \, ds\right)$

Two issues: • $\left\{ \int_0^t Y_s \, dB_s : t \geq 0 \right\}$ ... regularity?

• What Y's are integrable?

Ad! **Def** Given filtration $\{\mathcal{F}_t\}_{t\geq 0}$, a process $\{M_t : t \geq 0\}$ is a <u>martingale</u> if it is <u>adapted</u> (M_t is $\mathcal{F}_t$-meas.), <u>integrable</u> ($M_t \in L^1$) and $\forall t, s \geq 0$: $E(M_{t+s} \mid \mathcal{F}_t) = M_t$ a.s.

We say M is <u>continuous</u> if $t \mapsto M_t$ is continuous (everywhere)

<u>L^2-martingale</u> if $\forall t \geq 0 : M_t \in L^2$

Thm Let $\{\mathcal{F}_t\}_{t\geq 0}$ be Br. filtration for B = Br. motion. Suppose $\mathcal{F}_0$ contains all P-null events. Then $\forall Y \in \overline{\mathcal{D}_0}^{[\cdot]}$ there exists a continuous L^2-martingale $\{I_t : t \geq 0\}$ s.t.

$$\forall t \geq 0 : \quad P\left(I_t = \int_0^t Y_s \, dB_s \right) = 1$$

In particular, $\left\{ \int_0^t Y_s \, dB_s : t \geq 0 \right\}$ admits a cont. version.

Lemma (Doob's L^2-ineq.) Let $\{M_t : t \geq 0\}$ be a cont. L^2-martingale.
Then $\forall t \geq 0\ \forall \lambda > 0$: $P\left(\sup_{0 \leq s \leq t} |M_s| > \lambda\right) \leq \frac{1}{\lambda^2} E(M_t^2)$

Pf Let $0 = t_0 < t_1 < \dots < t_n \leq t$. Then $\{M_{t_i}\}_{i=0}^{n}$ is a discrete-time L^2-martingale
Doob's ineq. $P\left(\max_{i=0\dots n} |M_{t_i}| > \lambda\right) \leq \frac{1}{\lambda^2} E(M_{t_n}^2) \leq \frac{1}{\lambda^2} E(M_t^2)$
Exhausting $\mathbb{Q} \cap [0,t]$ we get (by MCT)

$$P\left(\sup_{\substack{0 \leq s \leq t \\ s \in \mathbb{Q}}} |M_s| > \lambda\right) \leq \frac{1}{\lambda^2} E(M_t^2)$$

Since $t \mapsto M_t$ is continuous, restriction to $s \in \mathbb{Q}$ can be dropped. $\boxtimes$

So $\|Y^{(k)} - Y\|_{L^2([0,t]\times\Omega)} \leq 2^{\lceil t \rceil} 2^{-k}$ if $k \geq \lceil t \rceil$
Define $I_t^{(k)} := \int_0^t Y^{(k)} dB_s$

Lemma: $\forall k \geq 1$: $I^{(k)}$ is continuous L^2-martingale

Pf: continuity & L^2-integrability clear from definition
NTS: martingale

i-th term in $\int_0^t Y^{(k)}\,dB_s$ leads to

$$E\left(Z_i\left(B_{(t+s)\wedge t_i} - B_{(t+s)\wedge t_{i-1}}\right)\middle|\mathcal{F}_t\right) \overset{?}{=} Z_i\left(B_{t\wedge t_i} - B_{t\wedge t_{i-1}}\right)$$

$\underline{t \le t_{i-1}}$: condition instead on $\mathcal{F}_{t_{i-1}}$ to get LHS $= 0$. RHS $= 0$

$\underline{t \ge t_i}$ conditioning moot & identify LHS $=$ RHS.

$\underline{t_{i-1} < t < t_i}$ $\quad B_{(t+s)\wedge t_i} - B_{(t+s)\wedge t_{i-1}} = B_{(t+s)\wedge t_i} - B_t + B_{t\wedge t_i} - B_{t\wedge t_{i-1}}$

Since $E\left(Z_i(B_{(t+s)\wedge t_i} - B_t)\middle|\mathcal{F}_t\right) \underset{t_{i-1}<t}{=} Z_i\, E\left(B_{(t+s)\wedge t_i} - B_t\middle|\mathcal{F}_t\right) = 0$

Plug 2^{nd} part and observe that 'it' is $\mathcal{F}_t$ - meas. $\boxtimes$

Continuing Pf of Thm:

$$P\left(\sup_{s\le t} |I_s^{(n)} - I_s^{(m)}| > 2^{-n/2}\right) \overset{\text{Doob}}{\le} \left(\frac{1}{2^{-n/2}}\right)^2 E\left(|I_t^{(n)} - I_t^{(m)}|^2\right)$$

$$\overset{\text{Itô}}{=} 2^n \|Y^{(n)} - Y^{(m)}\|^2_{L^2([0,t]\times\Omega)} \underset{m\ge n}{\le} 4\cdot 2^n \left(2^{-n+\lceil t\rceil}\right)^2 = 4^{1+\lceil t\rceil} 2^{-n}$$

By BC:

$$\Omega_0 = \bigcap_{N\ge 1}\left\{\sup_{s\le N} |I_s^{(n)} - I_s^{(n+1)}| > 2^{-n/2} \text{ i.o.}(n)\right\}^c$$

we get $\quad P(\Omega_0) = 1$

Define
$$I_t := \begin{cases} \lim_{n\to\infty} I_t^{(n)} & \text{on } \Omega_0 \\ 0 & \text{on } \Omega_0^c \end{cases}$$

Then $t \mapsto I_t$ is continuous. Since $\Omega_0^c \in \mathcal{F}_0$, I is adapted

Since $I_t^{(n)} \to \int_0^t Y_s \, dB_s$ in L^2 we get

$$P\left(I_t = \int_0^t Y_s \, dB_s\right) = 1$$

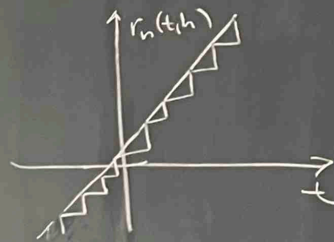
Also $I_t^{(n)} \xrightarrow{L^2} I_t$ and so martingale property carries over.

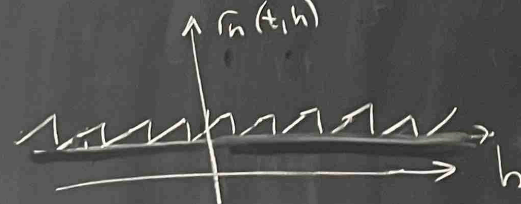
$$\boxed{\text{Thm } \mathcal{V}_0^{[\cdot,\cdot]} = \mathcal{V}, \text{ so all jointly-meas, adapted, locally square-integrable processes are Itô integrable}}$$

Pf idea Let $Y \in \mathcal{V}$. Approximate by piece-wise constant process by evaluating Y on "randomly shifted" dyadic grid.

Pf Let $Y \in \mathcal{D}$. Extend $Y_t := 0$ for $t<0$.

Define $r_n(t,h) := 2^{-n}\lfloor 2^n(t-h)\rfloor + h$





Note
- $t-2^{-n} \le r_n(t,h) \le t$
- $h \mapsto r_n(t,h)$ is 2^{-n}-per so $\int_0^1 f(r_n(t,h))\,dh = 2^n \int_0^{2^{-n}} f(\overset{t-h}{\cancel{r_n(t,h)}})\,dh$

Lemma $\forall T>0$: $E\left(\int_0^T dt \int_0^1 dh\, |Y_t - Y_{r_n(t,h)}|^2\right) \xrightarrow[n\to\infty]{} 0$

Pf Continuity of shift in L^2 implies

$$\int_0^T |Y_t(\omega) - Y_{\overset{t-h}{\cancel{r_n(t,h)}}}(\omega)|^2\,dt \xrightarrow[h\to 0]{} 0$$

LHS $\le 4\int_0^T Y_t(\omega)^2\,dt$, DCT tells us $E\left(\int_0^T |Y_t - Y_{\overset{t-h}{\cancel{r_n(t,h)}}}|^2\,dt\right) \xrightarrow[h\to 0]{} 0$

Now note: $E(\text{claim}) = \frac{1}{2^{-n}}\int_0^{2^{-n}} E(\quad)\,dh \xrightarrow[n\to\infty]{} 0$ $\boxed{\times}$

Find $n_k \to \infty$ s.t. $E\left(\int_0^T dt \int_0^1 dh \, |Y_t - Y_{r_{n_k}(t,h)}|^2\right) \le 4^{-k}$

Markov + Tonelli

$$\mathrm{Leb}\left(\left\{ h \in [0,1] : E\left(\int_0^T |Y_t - Y_{r_{n_k}(t,h)}|^2 dt\right) > 2^{-k} \right\}\right) \le \frac{4^{-k}}{2^{-k}} = 2^{-k}$$

BC: $\exists h \in [0,1] \ \forall T \ge 1 : \ E\left(\int_0^T |Y_t - Y_{r_{n_k}(t,h)}|^2 dt\right) \xrightarrow[k\to\infty]{} 0$

So $Y_t^{(k)} := \underbrace{Y_{r_{n_k}(t,h)}}_{\in \mathcal{V}_0} \mathbb{1}_{[0,k]}(t)$ obeys $[\![Y - Y^{(k)}]\!] \xrightarrow[k\to\infty]{} 0.$

$\boxtimes$