Stochastic integral via Itô isometry

previously: Itô integral $\int_0^t f(B_s)\,dB_s$ $\quad f \in C(\mathbb{R})$ (via Itô formula)
Paley-Wiener integral $\int h\, dW$ $=$ "$\int h(x)\, W(dx)$"
$h \in L^2(\mathcal{X}, \mathcal{G}, \mu)$
W = white noise

Today: Generalize to integral of form $\int_0^t Y_s\, dB_s$ where Y is a process s.t. Y_s depends only on $\{B_u : u \leq s\}$.

Def Let B = SBM on $(\Omega, \mathcal{F}, P)$. A family $\{\mathcal{F}_t\}_{t \geq 0}$ of σ-algebras on Ω are called Brownian filtration if

(1) $\forall\, 0 \leq s \leq t$: $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$

(2) B is adapted: $\forall t \geq 0$: B_t is $\mathcal{F}_t$-measurable

(3) B is Markov: $\forall t \geq 0$: $\sigma(B_{t+s} - B_t : s \geq 0) \perp\!\!\!\perp \mathcal{F}_t$

Ex $\mathcal{F}_t := \sigma(B_s : s \leq t)$, $\mathcal{F}_t^+ := \bigcap_{t' > t} \sigma(B_s : s \leq t')$, + add independent events

Def A stoch. process $\{Y_t : t \geq 0\}$ is <u>simple</u> if $\exists n \geq 0\ \exists\, 0 = t_0 < \dots < t_n\ \exists\, Z_0, \dots, Z_n \in L^\infty(\Omega, \mathcal{F}, P)$ s.t.

(1) $\forall t \geq 0$: $Y_t := \mathbb{1}_{\{0\}}(t) Z_0 + \sum_{i=1}^{n} Z_i \mathbb{1}_{(t_{i-1}, t_i]}(t)$

(2) $\forall i = 0 \dots n$: Z_i is $\mathcal{F}_{t_{i-1}}$-measurable $\quad (t_{-1} := 0)$

Here $(\Omega, \mathcal{F}, P)$ = prob. space, $\{\mathcal{F}_t\}_{t \geq 0}$ = filtration.

Note ... <u>simple</u> in prob. space but <u>step</u> in time!
(left-endpoint rule!)

Lemma Let Y be simple with representation as above. Then

$$\int_0^t Y_s \, dB_s := \sum_{i=1}^{n} Z_i \left(B_{t_i \wedge t} - B_{t_{i-1} \wedge t} \right)$$

$\min\{t_{i-1}, t\}$

does not depend on representation of Y. Moreover,

$$\forall t \geq 0\ \forall Y, \hat{Y} = \text{simple}\ \forall \alpha, \beta \in \mathbb{R} : \int_0^t (\alpha Y_s + \beta \hat{Y}_s) \, dB_s = \alpha \int_0^t Y_s \, dB_s + \beta \int_0^t \hat{Y}_s \, dB_s$$

Notation: $\mathcal{V}_0 :=$ set of simple processes (linear vector space)

Lemma: $\forall t \geq 0 \ \forall Y \in \mathcal{V}_0$: $E\left(\left[\int_0^t Y_s \, dB_s\right]^2\right) = E\left(\int_0^t Y_s^2 \, ds\right)$

(Itô isometry)

Pf $\text{LHS} = \sum_{i,j=1}^{n} E\Big(\underbrace{Z_i Z_j (B_{t_i \wedge t} - B_{t_{i-1} \wedge t})(B_{t_j \wedge t} - B_{t_{j-1} \wedge t})}_{T_{ij}}\Big)$

$$E(T_{ij}) \underset{i<j}{=} E\Big(Z_i Z_j (B_{t_i \wedge t} - B_{t_{i-1} \wedge t}) \underbrace{E\left(B_{t_j \wedge t} - B_{t_{j-1} \wedge t} \mid \mathcal{F}_{t_{j-1} \wedge t}\right)}_{=0}\Big)$$

$$\underset{i=j}{=} E\Big(Z_i^2 \underbrace{E\left((B_{t_i \wedge t} - B_{t_{i-1} \wedge t})^2 \mid \mathcal{F}_{t_{i-1} \wedge t}\right)}_{t_i \wedge t - t_{i-1} \wedge t}\Big) = E\left(\int_{t_{i-1} \wedge t}^{t_i \wedge t} Y_s^2 \, ds\right)$$

Now sum over $i = 1 \ldots n$. $\boxtimes$

Idea: Define a class of processes that can be reached from $\mathcal{V}_0$ by L^2-limits:

Def: Let $\mathcal{V} :=$ set of all $\{Y_s : s \geq 0\}$ that are:

(1) jointly measurable: $(\omega, t) \mapsto Y_t(\omega)$ is $\mathcal{F} \otimes \mathcal{B}[0,\infty)$-meas.

(2) adapted: $\forall t \geq 0$: Y_t is $\mathcal{F}_t$-meas.

(3) locally square integrable:

$$\forall t \geq 0: \ \|Y\|_{L^2([0,t] \times \Omega)} := \left[E \int_0^t Y_s^2 \, ds\right]^{1/2} < \infty$$

Lemma $\mathcal{V}_0 \subseteq \mathcal{V}$. Moreover, letting

$$[\![Y]\!] := \sum_{n \geq 1} 2^{-n} \left(\| Y \|_{L^2([0,n] \times \Omega)} \wedge 1 \right)$$

the map $Y, \tilde{Y} \mapsto [\![Y - \tilde{Y}]\!]$ is a pseudometric s.t.

$$[\![Y - \tilde{Y}]\!] = 0 \iff \mathrm{Leb} \otimes P \left(\{ (t,\omega) \in [0,\infty) \times \Omega : Y_t(\omega) \neq \tilde{Y}_t(\omega) \} \right) = 0$$

holds for all $Y, \tilde{Y} \in \mathcal{V}$.

Pf omitted.

Cor Let $\overline{\mathcal{V}_0}^{[\![\cdot]\!]} :=$ closure of $\mathcal{V}$ w.r.t. $[\![\cdot]\!]$.

Then $\forall t \geq 0$ the map $Y \mapsto \int_0^t Y_s \, dB_s$ extends continuously to all $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]}$. Moreover, we have

$$\forall Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]} \ \forall t \geq 0: \quad E\left(\left[\int_0^t Y_s \, dB_s \right]^2 \right) = E\left(\int_0^t Y_s^2 \, ds \right)$$

$$\forall Y, \hat{Y} \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]} \ \forall \alpha, \beta \in \mathbb{R} \ \forall t \geq 0: \quad \int_0^t (\alpha Y_s + \beta \bar{Y}_s) \, dB_s = \alpha \int_0^t Y_s \, dB_s + \beta \int_0^t \bar{Y}_s \, dB_s$$

a.s.

Pf Given $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]}$, there exist $\{Y^{(n)}\}_{n\geq 1} \in \mathcal{V}_0^{\mathbb{N}}$ s.t.

$$[\![Y - Y^{(n)}]\!] \xrightarrow[n \to \infty]{} 0$$

But then also

$$\forall t \geq 0 : E\left(\int_0^t (Y_s^{(n)} - Y_s)^2 ds \right) \longrightarrow 0$$

By Itô isometry $\{ \int_0^t Y_s^{(n)} dB_s \}_{n\geq 1}$ is Cauchy in L^2 and so converges in L^2. Limit r.v. is unique, denote it $\int_0^t Y_s \, dB_s$.

Now check the identities. $\boxtimes$

Q: What is $\overline{\mathcal{V}_0}^{[\![\cdot]\!]}$ relative to $\mathcal{V}$?

Lemma If $Y \in \mathcal{V}$ has left-continuous paths, then $Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]}$.

Pf Pick $Y \in \mathcal{V}$ left continuous. Assume first Y is bounded in sense: $\exists c > 0$: $\sup_{t \geq 0} |Y_t| \leq c$ a.s. Define:

$$Y_t^{(n)} := Y_0 \mathbb{1}_{\{0\}}(t) + \sum_{k=0}^{4^n} Y_{k2^{-n}} \mathbb{1}_{(k2^{-n}, (k+1)2^{-n}]}(t)$$

Left continuity: $\forall t \geq 0$: $Y_t^{(n)} \to Y_t$.

$$\text{BCT: } E\left(\int_0^t \left(Y_s^{(n)} - Y_s\right)^2 ds\right) \xrightarrow[n\to\infty]{} 0 \quad \forall t \geq 0$$

$$S_0 [\![Y^{(n)} - Y]\!] \xrightarrow[n\to\infty]{} 0 \quad \ldots \quad Y \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]}$$

Next assume $Y \in \mathcal{V}$ left cont. and set

$$\hat{Y}_t^{(M)} := Y_t \wedge M \vee (-M) \quad \ldots \text{ left cont., bdd} \Rightarrow Y^{(M)} \in \overline{\mathcal{V}_0}^{[\![\cdot]\!]}$$

$$\text{DCT: } E\left(\int_0^t \left(\hat{Y}_s^{(M)} - Y_s\right)^2 ds\right) \leq 4E\left(\int_0^t Y_s^2 \mathbb{1}_{|Y_s|>M}\, ds\right) \xrightarrow[M\to\infty]{\text{DCT.}} 0;$$

$\boxtimes$