White noise & Paley-Wiener integral

Last time: • Defined Itô integral $\int_0^t f(B_s)\,dB_s$ $(f \in C(\mathbb{R}))$
as the limit of left-endpoint Riemann-Stieltjes sums.

Today: Take this to more abstract level

Motivation: ODE from science: $\frac{dX_t}{dt} = a(t, X_t)$

Q: How to add noise? $\frac{dX_t}{dt} = a(t, X_t) + \underbrace{\dot{W}_t}_{\text{noise}}$

Want: (1) zero mean $E\dot{W}_t = 0$

(2) independence: $\forall s \neq t$: $\dot{W}_t \perp \dot{W}_s$

(3) $t \mapsto \dot{W}_t$ measurable

No such thing exists!

Solution:
- mollify the noise … (2) TRUE only if $|s-t| > \varepsilon$
- interpret ODE under integral … think of $\dot{W}$ as signed measure

Q: What natural signed measures with indep. structure do we have?

PPP(μ)

Poisson Point Process

Def Let $(\mathcal{X}, \Sigma, \mu)$ be meas. space, $\mu(\mathcal{X}) < \infty$. A Poisson point process with intensity μ is $\{N(A) : A \in \Sigma\}$ ($\mathbb{N} \cup \{0, \infty\}$-valued r.v.)

(1) $A \mapsto N(A)$ is countably additive ($N(\emptyset) = 0$)

(2) $\forall n \geq 1$ $\forall A_1 \ldots A_n \in \Sigma$ disjoint: $\{N(A_i) : i = 1 \ldots n\}$ are indep.

(3) $\forall A \in \Sigma$: $N(A) = \text{Poisson}(\mu(A))$

Lemma: Let $K = \text{Poisson}(\mu(\mathcal{X}))$, $\{X_i\}_{i \geq 1}$ iid $\sim \mu(\cdot)/\mu(\mathcal{X}) \perp K$.

Set
$$N(A) := \sum_{i=1}^{K} \mathbb{1}_A(X_i)$$

Then $\{N(A) : A \in \Sigma\}$ is PPP(μ).

Pf: $A \mapsto N(A)$ is a measure so count. additive.

$$P(N(A)=k) = \sum_{n\geq k} \overbrace{\frac{\mu(\mathcal{X})^n}{n!} e^{-\mu(\mathcal{X})}}^{P(K=n)} \underbrace{\binom{n}{k}\left(\frac{\mu(A)}{\mu(\mathcal{X})}\right)^k\left(1-\frac{\mu(A)}{\mu(\mathcal{X})}\right)^{n-k}}_{P\left(\sum_{i=1}^{n} 1_A(x_i)=k\right)}$$

$$= \ldots = \frac{\mu(A)^k}{k!} e^{-\mu(A)}$$

independence: $f : \mathcal{X} \to \mathbb{R}$ (bounded) ... $\int f \, dN$ meaningful $= \sum_{i=1}^{n} f(X_i)$

$$E\left(e^{i\int f\,dN}\right) = \sum_{n\geq 0} \frac{\mu(\mathcal{X})^n}{n!} e^{-\mu(\mathcal{X})} E\left(e^{if(X_1)}\right)^n$$

$$= \exp\left\{\mu(\mathcal{X})\left[E\left(e^{if(X_1)}\right)-1\right]\right\}$$

$$= \exp\left\{-\int \left(1-e^{if(x)}\right)\mu(dx)\right\}$$

Now take $\{A_i\}_{i=1}^{k}$ disjoint, $\lambda_1 \ldots \lambda_k \in \mathbb{R}$, $f(x) = \sum_{i=1}^{k} \lambda_i 1_{A_i}(x)$

Then $\int f \, dN = \sum_{i=1}^{k} \lambda_i N(A_i)$

$$\left(1-e^{-f(x)}\right) \overset{\text{disj}}{=} \sum_{j=1}^{k} \left(1-e^{i\lambda_j}\right) 1_{A_i}(x)$$

Put together:

$$E\left(e^{i\sum_{j=1}^{k}\lambda_j N(A_j)}\right) = \prod_{j=1}^{k} \exp\left\{-\left(1-e^{i\lambda_j}\right)\mu(A_j)\right\} \quad \boxtimes$$

ch.f. of Poisson$(\mu(A_j))$

Q: What natural signed measures with indep. structure do we have?

PPP(μ)

Poisson Point Process

Def Let $(\mathcal{X}, \Sigma, \mu)$ be meas. space, $\mu(\mathcal{X})<\infty$. A Poisson point process with intensity μ is $\{N(A): A\in\Sigma\}$ ($\mathbb{N}\cup\{0,\infty\}$-valued r.v.)

(1) $A\longmapsto N(A)$ is countably additive ($N(\emptyset)=0$)

(2) $\forall n\geq 1\ \forall A_1\dots A_n\in\Sigma$ disjoint: $\{N(A_i): i=1\dots n\}$ are indep.

(3) $\forall A\in\Sigma$: $N(A)\overset{d}{=}\text{Poisson}(\mu(A))$

- Compensated PPP $\quad \hat{N}(A) := N(A) - \mu(A)$
- Above construction makes sense even if μ on σ-finite.
- Compound PP: $f\in L^1(\mu)\longmapsto \int f\,dN$ (2^{nd} part of Lévy-Khinchin)
- Connection to PP from 1^{st} lecture:

$\mathcal{X}=[0,\infty)$, $\Sigma=\mathcal{B}([0,\infty))$, $\mu=$ Leb. measure

$N_t := N([0,t])$ RCLL

$\mathcal{X}=\mathbb{R}\times[0,\infty)$

$\mu=\nu\otimes\text{Leb}$

$Z_t := N(\mathbb{R}\times[0,t])$

White noise:

Def $(\mathcal{X}, \Sigma, \mu)$ = meas. space, $\mu(\mathcal{X}) < \infty$.

A (Gaussian) white noise is $\{W(A) : A \in \Sigma\}$ s.t.

$$\forall \{A_i\}_{i \geq n} \in \Sigma^{\mathbb{N}} \text{ disjoint: } \begin{cases} \{W(A_i)\}_{i \geq 1} \text{ indep} \\ W\left(\bigcup_{i \geq 1} A_i\right) \overset{a.s.}{=} \sum_{i \geq 1} W(A_i) \end{cases}$$

where sum converges a.s. & L^2.

and

$$\forall A \in \Sigma, \quad W(A) = N(0, \mu(A))$$

Note Another way to define this is by requiring that $\{W(A) : A \in \Sigma\}$ is multivariate normal with

$$\forall A, B \in \Sigma: \quad E(W(A)) = 0 \ \wedge \ E(W(A)W(B)) = \mu(A \cap B) \quad (*)$$

check it is a covariance!

Lemma: $(*) \Rightarrow \{W(A) : A \in \Sigma\}$ is white noise

Pf Consequence: if $\{W(A_i)\}_{i \geq 1}$ are indep centered normal, then

$$\sum_{i \geq 1} E(W(A_i)^2) = \sum_{i \geq 1} \mu(A_i) \underset{\{A_i\}_{i \geq 1} \text{ disj}}{=} \mu\left(\bigcup_{i \geq 1} A_i\right) \leq \mu(\mathcal{X}) < \infty.$$

By PW-theorem, $\sum_{i=1}^{\infty} W(A_i)$ conv. a.s. & L^2, ... rest as in HW1

Connection to SBM:

$\mathcal{X} = [0,\infty)$, $\Sigma = \mathcal{B}([0,\infty))$, $\mu = \text{Leb}$.

$B_t := W([0,t])$ ... then $\{B_t : t \geq 0\}$ has FDD of SBM ?

Paley-Wiener integral

Lemma Given f simple of the form $f = \sum_{i=1}^{n} \lambda_i 1_{A_i}$, we can define

$$\int f \, dW := \sum_{i=1}^{n} \lambda_i W(A_i)$$

Then $\int f \, dW$ does not depend on representation of f and obeys:

$$\boxed{E\left(\left(\int f \, dW\right)^2\right) = \int f^2 \, d\mu}$$

Pf. Indep. on repr. ... finite additivity of W (a.e.)
Rest ... calculation.

isometry $L^2(\mu) \longrightarrow L^2(\Omega)$. prob. space supporting W

Using isometry, we define $\int f \, dW$ for all $f \in L^2(\mu)$

Paley-Wiener integral