Itô integral via Itô formula

last time quadratic variation $V_t^{(2)}(B,\Pi) \xrightarrow[\|\Pi\|\to 0]{P} t$.

Lemma For a.e. B: $\forall t>0$ $\sup_{\Pi} V_t^{(2)}(B,\Pi) = +\infty$. Pf: HW3

Today: Attempt to derive "FTC" like representation for $f(B_t)-f(B_0)$.

Let $f \in C^2(\mathbb{R})$. Note Taylor's thm gives

$s=(1-\theta)x+\theta y$, $ds=(y-x)\,d\theta$

$y-s=(1-\theta)(y-x)$

$$f(y)-f(x) = f'(x)(y-x) + \int_x^y f''(s)(y-s)\,ds$$

$$= f'(x)(y-x) + (y-x)^2\int_0^1 f''((1-\theta)x+\theta y)(1-\theta)\,d\theta$$

Pick $\Pi=\{0=t_0<\dots<t_n=t\}$:

$$f(B_t)-f(B_0) = \sum_{i=1}^n \left[f(B_{t_i})-f(B_{t_{i-1}})\right]$$

$$= \sum_{i=1}^n f'(B_{t_{i-1}})(B_{t_i}-B_{t_{i-1}}) + \frac{1}{2}\sum_{i=1}^n f''(B_{t_{i-1}})(B_{t_i}-B_{t_{i-1}})^2$$

$$+ \sum_{i=1}^n \left[\int_0^1 \left(f''((1-\theta)B_{t_{i-1}}+\theta B_{t_i}) - f''(B_{t_{i-1}})\right)(1-\theta)\,d\theta\right](B_{t_i}-B_{t_{i-1}})^2$$

Q: What happens with these terms as $\|\Pi\| \to 0$?

Start with 2^{nd} sum:

$$\sum_{i=1}^{n} f''(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}})^2 = \sum_{i=1}^{n} f''(B_{t_{i-1}})(t_i - t_{i-1}) + \sum_{i=1}^{n} f''(B_{t_{i-1}})\left[(B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})\right]$$

Lemma For $f \in C^2(\mathbb{R})$, $t > 0$, Π = partition of $[0,t]$:

$$\underbrace{\sum_{i=1}^{n} f''(B_{t_{i-1}})\left[(B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})\right]}_{Q_n} \xrightarrow[\|\Pi\| \to 0]{P} 0$$

Pf: Fix $\varepsilon > 0$, let $M > 0$ be s.t. $P\left(\sup_{s \in [0,t]} |B_s| > M\right) < \varepsilon$ (cont. of B)

So $$P(|Q_n| > \varepsilon) \le P\left(\sup_{s \in [0,t]} |B_s| > M\right) + P\left(\Big|\underbrace{\sum_{i=1}^{n} C_i \left[(B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})\right]}_{\tilde{Q}_n}\Big| > \varepsilon\right)$$

where $C_i := \left(f''(B_{t_{i-1}}) \wedge \sup_{[-M,M]} f''\right) \vee \inf_{[-M,M]} f''$

$$P(|\tilde{Q}_n| > \varepsilon) \le \frac{1}{\varepsilon^2} E(\tilde{Q}_n^2) \le \frac{1}{\varepsilon^2}\left(\sup_{[-M,M]} |f''|\right)^2 \sum_{i=1}^{n} \mathrm{Var}\left((B_{t_i} - B_{t_{i-1}})^2\right)$$

$$= 3\sum_{i=1}^{n} |t_i - t_{i-1}|^2 \le 3t\,\|\Pi\|$$

Take $\|\Pi\| \to 0$ followed by $\varepsilon \downarrow 0$ $\boxtimes$

Cor. For $f \in C^2(\mathbb{R})$, $t>0$, Π = partition of $[0,t]$:

$$\sum_{i=1}^{n} f''(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}})^2 \xrightarrow[\|\Pi\| \to 0]{P} \int_0^t f''(B_s)\,ds$$

Lemma For $f \in C^2(\mathbb{R})$, $t>0$, Π = part. of $[0,t]$:

$$F(\Pi) \xrightarrow[\|\Pi\| \to 0]{P} 0$$

Pf Denote

$$\operatorname{osc}_h(A, \delta) := \sup\{ |h(x) - h(y)| : x, y \in A \wedge |x-y| < \delta \}$$

Then $|F(\Pi)| \leq \frac{1}{2} \operatorname{osc}_{f''}\left(B([0,t]), \operatorname{osc}_B([0,t], \|\Pi\|) \right) V_t^{(2)}(B, \Pi)$.

By cont. of f'' & B, the oscillation $\xrightarrow[\|\Pi\| \to 0]{} 0$

Since $V_t^{(2)}(B, \Pi)$ is bounded in prob., we get the claim ☒

Corollary There exists a r.v. $I_t(f')$ s.t. $\forall t>0$ $\forall \Pi$ = partitions of $[0,t]$

$$\sum_{i=1}^{n} f'(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}}) \xrightarrow[\|\Pi\| \to 0]{P} I_t(f')$$

assuming $f \in C^2(\mathbb{R})$

Applying this to antiderivative of f, we have

Denoting $\quad I_t(f, \Pi) := \sum_{i=1}^{n} f(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}})$

Then $\forall f \in C^1(\mathbb{R})$ $\forall t > 0$: $\quad I_t(f, \Pi) \xrightarrow[\|\Pi\| \to 0]{P} I_t(f)$

where $I_t(f)$ is a r.v. (indep. of seq. of partition)

Notation: $I_t(f) =: \int_0^t f(B_s)\,dB_s$

<u>Lemma</u> $\forall f \in C_b(\mathbb{R})$ $\forall t > 0$ $\exists I_t(f)$ s.t.

$$I_t(f, \Pi) \xrightarrow[\|\Pi\| \to 0]{P} I_t(f)$$

(continuity sufficient)

<u>Pf</u>: Fix $\varepsilon > 0$, let $P(\sup_{[0,t]} |B| > M) < \varepsilon$. (Weierstass)

For $f \in C_b(\mathbb{R})$ find $f_\varepsilon \in C^1(\mathbb{R}) \cap C_b(\mathbb{R})$ s.t. $\sup_{[-M, M]} |f - f_\varepsilon| < \varepsilon$.

Further detail ... notes

As a consequence, we have two theorems:

Thm $\forall f \in C(\mathbb{R})\ \forall t>0\ \exists\, I_t(f) =: \int_0^t f(B_s)\,dB_s$

s.t. $$I_t(f,\Pi) \xrightarrow[\|\Pi\|\to 0]{P} \int_0^t f(B_s)\,dB_s$$ (defines Itô integral)

Thm $\forall f \in C^2(\mathbb{R})\ \forall t>0$:

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)\,dB_s + \underbrace{\frac{1}{2}\int_0^t f''(B_s)\,ds}_{\text{Itô correction}}$$

(Itô formula).

Note $\int_0^t f(B_s)\,dB_s$ is NOT ordinary Stieltjes integral

E.g. $f(x)=x$ $$\sum_{i=1}^n B_{t_i}(B_{t_i}-B_{t_{i-1}}) - \sum_{i=0}^n B_{t_{i-1}}(B_{t_i}-B_{t_{i-1}})$$

$$= \sum_{i=1}^n (B_{t_i}-B_{t_{i-1}})^2 = V_t^{(2)}(B,\Pi) \xrightarrow[\|\Pi\|\to 0]{P} t$$

right endpoint − left endpoint

What other approximations?

$$I_t^{(\theta)}(f,\Pi) = \sum_{i=1}^{n} f\left((1-\theta)B_{t_{i-1}} + \theta B_{t_i}\right)(B_{t_i} - B_{t_{i-1}}) \quad (f \in C^1)$$

Thm $\quad I_t^{\theta}(f,\Pi) \xrightarrow[\|\Pi\|\to 0]{P} \int_0^t f(B_s)\,dB_s + \theta \int_0^t f'(B_s)\,ds$

$\theta = 1/2$ Stratonovich integral $\int_0^t f(B_s) \circ dB_s$