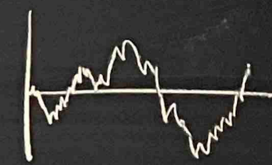
Sample-path properties of SBM

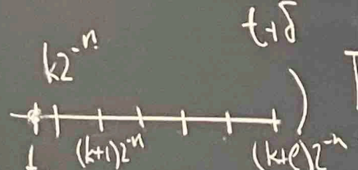
Thm (Paley, Wiener, Zygmund 1933) Let $\gamma > 1/2$.
Then for a.e. sample $B = \{B_t : t \geq 0\}$ of SBM:

$$\forall t \geq 0 : \limsup_{s \downarrow t} \frac{|B_t - B_s|}{|t-s|^\gamma} = +\infty.$$

In particular, a.e. path of SBM is nowhere differentiable.



Pf **Idea:** If t is γ-Hölder point, then in $\ell \geq 1$ consecutive intervals to the right of t the path changes very little.



Precisely Assume $t \geq 0$ is s.t. $\exists \delta > 0\ \exists C < \infty\ \forall s \in (t, t+\delta)$: $\frac{|B_t - B_s|}{|t-s|^\gamma} \leq C$
Take $\ell \geq 1$ s.t. $\ell(\gamma - 1/2) > 1$. For $n \geq 1$ set $k := \lceil t2^n \rceil$
and note for $j = 1, \ldots, \ell$:

$$\left|B_{(k+j)2^{-n}} - B_{(k+j-1)2^{-n}}\right| \leq \left|B_{(k+j)2^{-n}} - B_t\right| + \left|B_{(k+j-1)2^{-n}} - B_t\right| \leq 2C\left((\ell+1)2^{-n}\right)^\gamma$$
$$= 2C(\ell+1)^\gamma 2^{-n\gamma}$$

Denoting $A^{(m)}_{n,k,j} := \{ | B_{(k+j)2^{-n}} - B_{(k+j-1)2^{-n}} | \leq m 2^{-n\gamma} \}$

Then $\left\{ \exists t \in [0, 1/2] : \limsup_{s \downarrow t} \frac{|B_t - B_s|}{|t-s|^\gamma} < \infty \right\}$

$$\subseteq \bigcup_{m \geq 1} \left\{ \bigcup_{k=1}^{2^n} \bigcap_{j=1}^{\ell} A^{(m)}_{n,k,j} \ \text{i.o.}\ (n) \right\}$$

Now estimate:

$$P(A^{(m)}_{n,k,j}) = P(|N(0, 2^{-n})| \leq m 2^{-n\gamma}) = P(|N(0,1)| \leq m 2^{-n(\gamma - 1/2)})$$

$$\leq 2m 2^{-n(\gamma - 1/2)}$$

Using indep. of increments & union bound:

$$P\left(\bigcup_{k=1}^{2^n} \bigcap_{j=1}^{\ell} A^{(m)}_{n,k,j} \right) \leq 2^n \left(2m 2^{-n(\gamma - 1/2)} \right)^{\ell} = (2m)^{\ell} 2^{-n(\ell(\gamma - 1/2) - 1)}$$

Summable on n !

Borel-Cantelli: $P(\text{RHS}) = 0$

Taking unions over intervals proves claim on all of $[0, \infty)$. $\boxtimes$

Q: What's the true regularity?

<u>Thm</u> (Law of iterated logarithm, Khinchin '33)

$$\forall t \geq 0: \quad \limsup_{s \downarrow t} \frac{|B_t - B_s|}{\sqrt{|t-s| \log\log \frac{1}{|t-s|}}} = \sqrt{2} \quad \text{a.s.}$$

<u>Cor</u>

$$\limsup_{t \to \infty} \frac{|B_t|}{\sqrt{t \log\log t}} = \sqrt{2} \quad \text{a.s.}$$

<u>Pf</u> $W_t := t B_{1/t}$ is also SBM (up to null)

<u>Thm</u> (Lévy modulus of continuity '37)

$$\limsup_{\delta \downarrow 0} \sup_{\substack{0 \leq s < t \leq 1 \\ t-s < \delta}} \frac{|B_t - B_s|}{\sqrt{\delta \log 1/\delta}} = \sqrt{2} \quad \text{a.s.}$$

points with such oscillation are called <u>fast points</u>

Fractal structure of fast points studied as well.

Thm (Dvoretzky '63, Davis '83)

$$\inf_{t\in[0,1]} \limsup_{s\to t} \frac{|B_t - B_s|}{\sqrt{|t-s|}} = 1 \qquad \text{slow points}$$

Take away message: SBM is $\sim \frac{1}{2}$-Hölder
so too rogh for usual
argumets from ODE, Stieljes integral, ...

Def Given $t>0$, $\Pi = \{t_i\}_{i=0}^{n}$ partition of $[0,t]$ $\quad 0 = t_0 < t_1 < \cdots < t_n = t$
and $f : [0,t] \to \infty$, define

$$V_t^{(p)}(f, \Pi) := \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p$$

This is p-variation of f for partition Π. $\quad (p>0)$

$$E\, V_t^{(p)}(B,\Pi) = E\Big(\sum_{i=1}^n (B_{t_i} - B_{t_{i-1}})^p\Big)$$

$$= \underbrace{E(|N(0,1)|^p)}_{C} \sum_{i=1}^n |t_i - t_{i-1}|^{p/2} \begin{cases} = Ct & p=2 \\ \le Ct\|\Pi\|^{\frac{p}{2}-1} & p>2 \end{cases}$$

So $p=2$ relevant for SBM !

$$\mathrm{Var}\big(V_t^{(2)}(B,\Pi)\big) = \sum_{i=1}^n \mathrm{Var}\big((B_{t_i} - B_{t_{i-1}})^2\big)$$

$$= \underbrace{\mathrm{Var}(|N(0,1)|^2)}_{\tilde{C}=3} \sum_{i=1}^n |t_i - t_{i-1}|^2 \le \tilde{C}\, t\, \|\Pi\|$$

<u>Lemma</u> (1) $\|\Pi_n\| \to 0 \Rightarrow V_t^{(2)}(B,\Pi_n) \xrightarrow[n\to\infty]{P} t$

(2) $\sum_{n\ge 1} \|\Pi_n\| < \infty \Rightarrow V_t^{(2)}(B,\Pi_n) \xrightarrow[n\to\infty]{} t$ a.s.

(3) if $\{\Pi_n\}$ are nested $\wedge$ $\|\Pi_n\| \to 0$

$\Rightarrow V_t^{(2)}(B,\Pi_n) \xrightarrow[n\to\infty]{} t$ a.s.

Pf of (1,2): $P\left(|V_t^{(2)}(B,\Pi) - t| > \varepsilon \right) \le \frac{1}{\varepsilon^2} \mathrm{Var}\left(V_t^{(2)}(B,\Pi) \right)$

$$\le \frac{1}{\varepsilon^2} \tilde{C} t \, \|\Pi\|$$

Pf of (3) notes. $\boxtimes$

Note Limit non-random by accident:

Lemma Let $f \in C^1(\mathbb{R})$. Then $\forall t \ge 0$:

$$V_t^{(2)}(f \circ B, \Pi) \xrightarrow[\|\Pi\| \to 0]{P} \int_0^t f'(B_s)^2 \, ds$$

Pf HW idea: $\text{LHS} = \sum_{i=1}^n \left(f(B_{t_i}) - f(B_{t_{i-1}}) \right)^2$

$$= \sum_{i=1}^n f'(B_{t_{i-1}})^2 (B_{t_i} - B_{t_{i-1}})^2 + \text{error}$$

$$= \sum_{i=1}^n f'(B_{t_{i-1}})^2 (t_i - t_{i-1}) + \sum_{i=1}^n f'(B_{t_{i-1}})^2 \left[(B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1}) \right] + \text{error}$$