Uniqueness & properties of SBM

last time: constructed a SBM, continuity necessary for uniqueness.

idea: view $t \longmapsto B_t$ as function-valued RV.

Denote: $C[0,\infty) := \{ f \in \mathbb{R}^{[0,\infty)} : \text{continuous} \}$

$$\rho(f,g) := \sum_{n \geq 1} 2^{-n} \sup_{t \in [0,n]} |f(t) - g(t)| \wedge 1 .$$ ... metric of loc.-uniform convergence

Lemma ρ is a metric on $C[0,\infty)$ that makes $(C[0,\infty), \rho)$ complete & separable.
In particular, $\exists \{f_n\}_{n \geq 1} \in C[0,\infty)^{\mathbb{N}}$ s.t. for $U_k(f,a) := \{ g \in C[0,\infty) : \sup_{t \in [0,k]} |f(t) - g(t)| < a \}$
every open set in $C[0,\infty)$ is a countable union of finite intersections of $\{ U_k(f_n, a) : n,k \geq 1,\ a \in \mathbb{Q}^+ \}$.

Pf. HW2

We call $(C[0,\infty), \mathcal{B}(C[0,\infty)))$ ... the canonical/Wiener space

Lemma Given $\{X_t : t \in [0,\infty)\}$ whose every path is continuous, the map $t \longmapsto X_t$ defines a $C[0,\infty)$-valued RV.

Let $(\Omega, \mathcal{F})$ = space supporting X.

Pf: $$X^{-1}(U_k(f,a)) = \bigcup_{\substack{a' \in \mathbb{Q}^+ \\ a' < a}} \bigcap_{t \in [0,k] \cap \mathbb{Q}} \underbrace{\{|X_t - f(t)| < a'\}}_{\in \mathcal{F}} \in \mathcal{F}$$

By Lemma, $X^{-1}(O) \in \mathcal{F}$ for all $O \subseteq C[0,\infty)$ open.

Now use $\{A \in \mathcal{B}(C[0,\infty)) : X^{-1}(A) \in \mathcal{F}\}$ is σ-alg.

this σ-alg then coincides with $\mathcal{B}(C[0,\infty))$. $\boxtimes$

Cor An $\mathbb{R}$-valued $\{X_t : t \in [0,\infty)\}$ whose every path is continuous induces a prob. measure $P^{(X)}$ on $(C[0,\infty), \mathcal{B}(C[0,\infty)))$ via

$$P^{(X)}(A) := P(X \in A)$$

This is the <u>distribution of X</u>.

Q: What determines $P^{(X)}$?

Lemma Let $\mathcal{P}$ = π-system s.t. $\sigma(\mathcal{P}) = \mathcal{B}(\mathbb{R})$. Every prob. measure on $(C[0,\infty), \mathcal{B}(C[0,\infty)))$ is determined by its values on sets

$$\bigcap_{i=1}^{n} \{f \in C[0,\infty) : f(t_i) \in A_i\} : n \geq 1,\ t_1 \ldots t_n \in [0,\infty),\ A_1 \ldots A_n \in \mathcal{P}$$

In particular, μ is determined by its FDD's.

Pf: HW2

Cor Every SBM induces same distribution on Wiener space.

Sample path properties

Prop: Let $\{B_t : t \geq 0\}$ = SBM. Then so is $\{W_t : t \geq 0\}$ defined for $t \geq 0$ by:

(1) $W_t := B_{t+s} - B_s$, for $s \geq 0$ fixed (time shift)

(2) $W_t := \frac{1}{a} B_{ta^2}$ for $a \neq 0$ fixed. (diffusive scaling)

(3) $W_t := \begin{cases} t B_{1/t} & t > 0 \\ 0 & t = 0 \end{cases}$ on $\{\lim_{t \downarrow 0} t B_{1/t} = 0\}$ and $W_t := 0$ else. (time inversion)

Pf (1,2) checked directly

For (3) use that $\{B_t, t\ge 0\}$ is multivariate normal with $EB_t=0$, $Cov(B_t,B_s)=t\wedge s$.

Let $\hat W_t := \begin{cases} t B_{1/t} & t>0 \\ 0 & t=0 \end{cases}$. Then W —— " —— with $E\tilde W_t=0$ (HW!)

So $\tilde W$ has FDD's of SBM. Need to address continuity. $Cov(\tilde W_t,\tilde W_s) \underset{t,s>0}{=} ts\,\frac{1}{t}\wedge\frac{1}{s} = t\wedge s$

$$P\left(\lim_{t\downarrow 0}\tilde W_t=0\right) \underset{\substack{\text{using cont.}\\ \text{for } t>0}}{=} P\left(\bigcap_{n\ge1}\bigcup_{m\ge1}\bigcap_{k\ge m}\left\{\sup_{t\in\mathbb{Q}\cap[0,1/k]}|\tilde W_t|<\frac{1}{n}\right\}\right)$$

$$\underset{\tilde W \overset{FDD}{=} B}{=} P(\hat W \leftrightarrow B) = P\left(\lim_{t\downarrow 0} B_t=0\right)=1$$

So W is a version of $\tilde W$ that has cont. paths. ☒

Cor (SLLN for SBM) $\lim_{t\to\infty}\frac{B_t}{t}=0$ a.s.

Pf $\frac{1}{t}B_t = \tilde W_{1/t}$ $\quad W_s \xrightarrow[s\downarrow 0]{} 0$ a.s. so $\tilde W_{1/t}\xrightarrow[t\to\infty]{} 0$ a.s. ☒

Previous proof

Note Additional symmetries pop up for d-dimensional SBM $(\underbrace{B_t^{(1)},\dots,B_t^{(d)}}_{\text{iid SBM}})$

- global rotation invariance
- conformal invariance (global)
- in d=2, also local conformal maps