Brownian local time

Interested in additive functionals $t \mapsto \int_0^t f(X_s)\,ds$ ← $\mathbb{R}^d$-valued

Def $\mathcal{L}_t^X(A) := \int_0^t 1_A(X_s)\,ds$

Then $\int_0^t f(X_s)\,ds = \int f(x)\,\mathcal{L}_t^X(dx)$ (monotone class argument)

We call $\mathcal{L}_t^X$ = occupation time measure

- $A \mapsto \mathcal{L}_t^X(A)$ is Borel measure on $\mathbb{R}^d$, $\mathcal{L}_t^X(\mathbb{R}^d) = t$
- additivity: $\mathcal{L}_{t+s}^X(A) = \underbrace{\mathcal{L}_t^X(A) + \mathcal{L}_s^{\theta_t(X)}(A)}_{\text{indep if } X = \text{SBM.}}$

For 1-d SBM, Tanaka formula tells us

$$\frac{1}{2\varepsilon}\mathcal{L}_t^B\big((x-\varepsilon, x+\varepsilon)\big)$$

$$= \frac{1}{2\varepsilon}\int_0^t 1_{|B_s - x| < \varepsilon}\,ds \xrightarrow[\varepsilon \downarrow 0]{P} |B_t - x| - |B_0 - x| - \int_0^t \operatorname{sgn}(B_s - x)\,dB_s$$

$$= \cancel{\tfrac{1}{2}}\,2\left[(B_t - x)_+ - (B_0 - x)_+ + \int_0^t 1_{(-\infty, x]}(B_s)\,dB_s\right]$$

$|z| = 2z_+ - z$

$\operatorname{sgn}(z - x) = 1 - 2\,1_{(-\infty, x]}(z)$

Key issue: prove this limit simultaneously for all $x \in \mathbb{R}$.

Thm (Trotter 1954) There exists a continuous process $\{I(t,x): t\geq 0, x\in\mathbb{R}\}$ s.t. $\forall t\geq 0\ \forall x\in\mathbb{R}$: $I(t,x) = \int_0^t 1_{(-\infty,x]}(B_s)\,dB_s$ P^0-a.s.

Moreover, setting

$$L_t(x) = \frac{1}{2}\left[(B_t-x)_+ - (B_0-x)_+ + I(t,x)\right]$$

Then $\exists\, \Omega^* \in \mathcal{F}$, $P(\Omega^*) = 1$, s.t.

$$\forall A\in\mathcal{B}(\mathbb{R})\ \forall t\geq 0:\ \mathcal{L}_t^B(A) = \int_A L_t(x)\,dx \quad \text{on } \Omega^*$$

Pf Idea:
- construct the cont version of stoch integral using Kolmogorov-Čentsov
- check formula for $\mathcal{L}^t(A)$ works.

Lemma (Burkholder-Davies-Gundy inequality)

$\forall n\geq 1\ \exists A_n\in(0,\infty)\ \forall M\in\mathcal{M}^{loc}_{cont}\ \forall t\geq 0$:

$$E(M_t^{2n}) \leq A_n E(\langle M\rangle_t^n)$$

Pf Notes $n=2$

Pick $0 \le u < t$, $x < y$:

$$E\left(\left|\int_0^t 1_{(-\infty,y]}(B_s)\,dB_s - \int_0^u 1_{(-\infty,x]}(B_s)\,dB_s\right|^{2n}\right)$$

$$= E\left(\left|\int_u^t 1_{(-\infty,y]}(B_s)\,dB_s + \int_0^u 1_{(x,y]}(B_s)\,dB_s\right|^{2n}\right)$$

$(a+b)^{2n} \le 4^n(a^{2n}+b^{2n})$

$$\le 4^n E\left(\left|\int_u^t 1_{(-\infty,y]}(B_s)\,dB_s\right|^{2n}\right) + 4^n E\left(\left|\int_0^u 1_{(x,y]}(B_s)\,dB_s\right|^{2n}\right)$$

$$\overset{\text{BDG}}{\le} 4^n A_n E\left(\underbrace{\left|\int_u^t 1_{(-\infty,y]}(B_s)\,ds\right|^n}_{\le |t-u|^n}\right) + 4^n A_n E\left(\left|\int_0^u 1_{(x,y]}(B_s)\,ds\right|^n\right)$$

2^{nd} expectation $$= n! \int_{0 \le s_1 < s_2 < \dots < s_n \le t} \prod_{i=1}^n \left[f_{s_i - s_{i-1}}(x_i - x_{i-1})\, 1_{(x,y]}(x_i) \right] ds_1 \dots ds_n\, dx_1 \dots dx_n$$

$$f_s(x) = \frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}} \le \frac{1}{\sqrt{2\pi s}}$$

$$\le n!\,|y-x|^n \int_{0 \le s_1 < \dots < s_n \le t} \prod_{i=1}^n \frac{1}{\sqrt{2\pi(s_i - s_{i-1})}}\, ds_1 \dots ds_n$$

$$\le n!\,|y-x|^n \left(\int_0^t \frac{1}{\sqrt{2\pi s}}\, ds\right)^n$$

Hence: expectation above $$\le A_n 4^n |t-u|^n + A_n 4^n n! \left(\int_0^t \frac{1}{\sqrt{2\pi s}}\, ds\right)^n |x-y|^n$$

$$\le A_n' \,\|(u,x) - (t,y)\|_2^n$$

$\underline{KC}$: $E(|Z_{u,x} - Z_{t,y}|^{\alpha}) \leq C \,\|(u-t, x-y)\|^{2+\beta}$ $(d=2)$

$\Rightarrow$ $u,x \mapsto Z_{u,x}$ admits γ-Hölder cont. version with $\gamma < \frac{\beta}{\alpha}$.

$\alpha = 2n$

$\beta = n-2$

$\gamma < \frac{n-2}{2n}$

$\Rightarrow$ $t,x \mapsto \int_0^t 1_{(-\infty,x]}(B_s)\,dB_s$

admits locally γ-Hölder cont. version

for every $\gamma < 1/2$.

Call this version $I(t,x)$

$\underline{Next}$ Take $f \in C_c^2(\mathbb{R})$. Then Itô formula tells us

$$\int_{\mathbb{R}} \mathcal{L}_t^B(dx) f''(x) = \int_0^t f''(B_s)\,ds$$

$$\overset{\text{Itô}}{=} 2\left[f(B_t) - f(B_0) - \int_0^t f'(B_s)\,dB_s \right]$$

Now
$$\int_0^t f'(B_s)\,dB_s = -\int_0^t \left(\int_{\mathbb{R}} f''(y)\, \overset{1_{[B_s,\infty)}(y)}{1_{(-\infty,y]}(B_s)}\, dy \right) dB_s$$

$$\overset{\text{Fubini}}{=} -\int_{\mathbb{R}} f''(y) \Big(\underbrace{\int_0^t 1_{(-\infty,y]}(B_s)\,dB_s}_{I(t,y)} \Big)\, dy$$

$$= -\int_{\mathbb{R}} f''(y) \tfrac{1}{2} L_t(x) + \underbrace{\int f''(y)\,(B_t - y)_+\,dy}_{= f(B_t)} - \underbrace{\int f''(y)(B_0 - y)_+\,dy}_{f(B_0)}$$

We conclude:

$$\forall f \in C_c(\mathbb{R}): \quad \int_{\mathbb{R}} \mathcal{L}_t^B(dx)\, f(x) = \int L_t(x)\, f(x)\, dx \quad P^0\text{-a.s.}$$

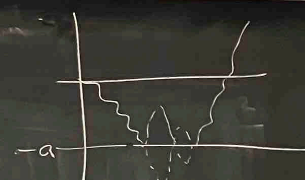
$f \uparrow \mathbb{1}_{(a,b)}$

$$\Rightarrow \forall\, a<b: \quad \underbrace{\mathcal{L}_t^B((a,b)) = \int_a^b L_t(x)\, dx \quad P^0\text{-a.s.}}_{=:\, \Omega_{a,b}}$$

Set $\Omega^* := \bigcap_{\substack{a,b \in \mathbb{Q} \\ a<b}} \Omega_{a,b}$, then $P(\Omega^*) = 1$, So π-λ-theorem tells us

$$\forall A \in \mathcal{B}(\mathbb{R}): \quad \mathcal{L}_t^B(A) = \int_A L_t(x)\, dx. \qquad \boxtimes$$

<u>Application:</u> SDE for reflected B.M.:



$$X_t := |B_t + a| - a$$

Tanaka:

$$dX_t = \underbrace{\operatorname{sgn}(B_t + a)}_{=1}\, dB_t + dL_t(-a)$$

<u>Extends</u> (by Tanaka) to SBM in convex domains in $\mathbb{R}^d$

Thm (Generalized Itô) Let $f \in C(\mathbb{R})$ s.t. $D^+f(x) := \lim_{h \downarrow 0} \frac{f(x+h) - f(x)}{h}$ exists with $x \longmapsto D^+f(x)$ RC. Assume also that D^+f is BV, and let μ be the signed measure s.t.

$$\forall a<b: \quad \mu((a,b]) = D^+f(b) - D^+f(a).$$

Then $\forall t \geq 0$:

$$f(B_t) = f(B_0) + \int_0^t D^+f(B_s)\,dB_s + \frac{1}{2}\int_{\mathbb{R}} L_t(x)\,\mu(dx)$$

Cor Standard Itô formula holds when $f \in C^1(\mathbb{R}) \wedge f' \in AC$.

Pf $\mu((a,b]) = f'(b) - f'(a) = \int f''(x)\,dx$

then $\int L_t(x)\,\mu(dx) = \int L_t(x) f''(x)\,dx$

$$= \int_0^t f''(B_s)\,ds$$

$\boxtimes$